System Dynamics

Second Edition

William J. Palm III
University of Rhode Island

Higher Education

Boston Burr Ridge, IL Dubuque, IA New York San Francisco St. Louis
Bangkok Bogotá Caracas Kuala Lumpur Lisbon London Madrid Mexico City
Milan Montreal New Delhi Santiago Seoul Singapore Sydney Taipei Toronto

9.2 RESPONSE OF SECOND-ORDER SYSTEMS

The equations of motion of many systems containing mass, spring, and damping elements have the form

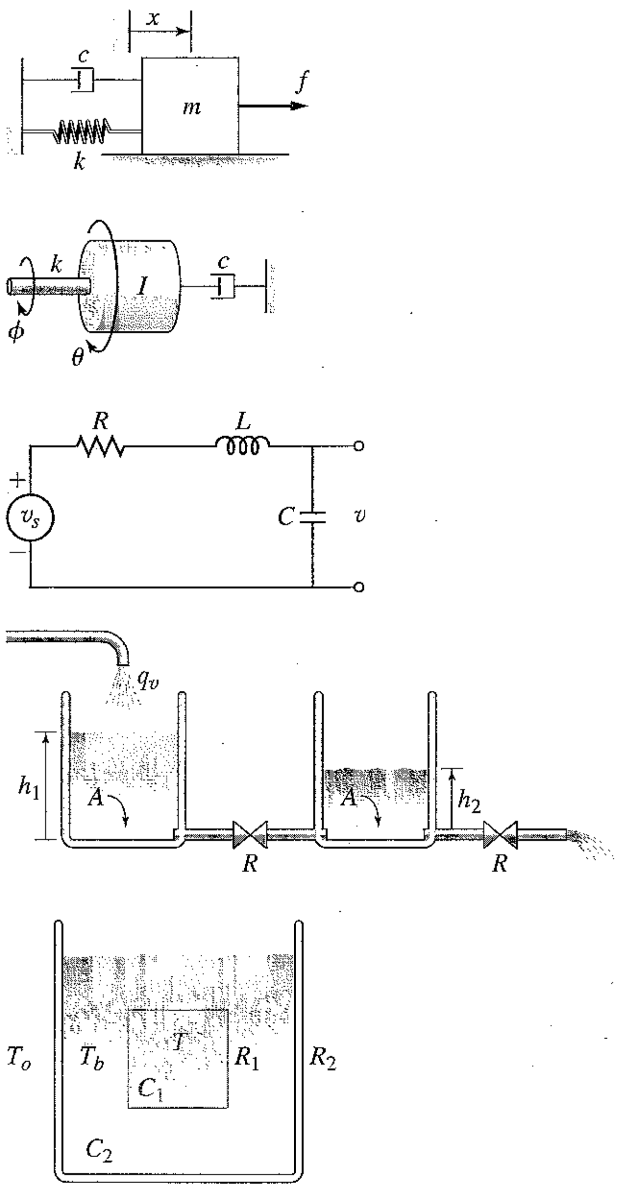
$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{9.2.1}$$

where $f(t)$ is the input. Its transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{9.2.2}$$

Figure 9.2.1 shows examples of other types of systems that have the same model form. The solution of this equation, and therefore the form of the free and forced responses, depends on the values of the two characteristic roots, obtained from the characteristic

Figure 9.2.1 Some second-order systems.



(a) $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

(b) $$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$

(c) $$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

(d) $$RA\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_v$$

$$RA\frac{dh_2}{dt} + g(h_2 - h_1) + gh_2 = 0$$

(e) $$R_1C_1\frac{dT}{dt} + T = T_b$$

$$R_1R_2C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b$$

$$= R_2T + R_1T_o$$

equation $ms^2 + cs + k = 0$. Recall from the discussion in Chapter 3 that this model is stable if both of its roots are real and negative or if the roots are complex with negative real parts. This is true if m, c, and k have the same sign.

A related model form is

$$m\ddot{x} + c\dot{x} + kx = a\dot{g}(t) + bg(t) \tag{9.2.3}$$

where $g(t)$ is the input. Its transfer function is

$$\frac{X(s)}{G(s)} = \frac{as + b}{ms^2 + cs + k} \tag{9.2.4}$$

So this model has numerator dynamics. It is important to understand that the input does not affect the characteristic equation, and therefore does not affect the stability of the model or its free response. Thus (9.2.1) and (9.2.3) have the same stability characteristics and the same free response, because they have the same characteristic equation, $ms^2 + cs + k = 0$.

The formulas to be developed in this section are based on the transfer function model form. Models in state variable form can always be reduced to transfer function form. For example, consider the model of an armature-controlled dc motor, developed in Section 6.4.

$$L_a \frac{di_a}{dt} = v_a - i_a R_a - K_b \omega \tag{9.2.5}$$

$$I \frac{d\omega}{dt} = K_T i_a - c\omega - T_L \tag{9.2.6}$$

This is a second-order linear model and it can be reduced to the forms of (9.2.2) and (9.2.4), as shown in Section 6.4. The results are

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a I s^2 + (R_a I + cL_a)s + cR_a + K_b K_T} \tag{9.2.7}$$

which has the form of (9.2.2), and

$$\frac{I_a(s)}{V_a(s)} = \frac{Is + c}{L_a I s^2 + (R_a I + cL_a)s + cR_a + K_b K_T} \tag{9.2.8}$$

which has the form of (9.2.4).

UNDAMPED RESPONSE

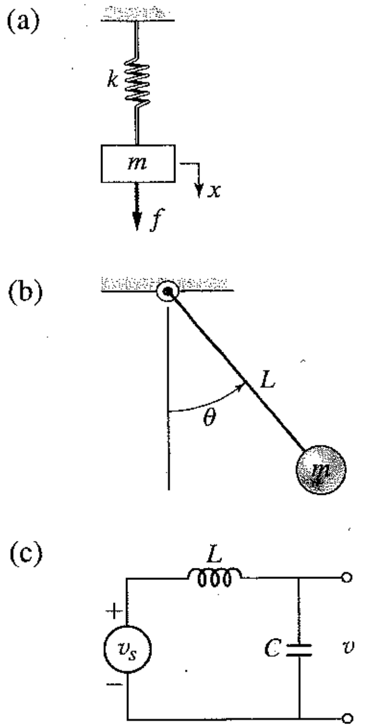
Consider the undamped systems shown in Figure 9.2.2. They all have the same model form: $m\ddot{x} + kx = f(t)$. In the first system, suppose $f(t) = 0$ and we set the mass in motion at time $t = 0$ by pulling it to a position $x(0)$ and releasing it with an initial velocity $\dot{x}(0)$. From Case 3 in Table 9.2.1, the response has the form $x(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t$, where we have defined

$$\omega_n = \sqrt{\frac{k}{m}} \tag{9.2.9}$$

Using the initial conditions we find that the constants are $C_1 = \dot{x}(0)/\omega_n$ and $C_2 = x(0)$. Thus the solution is

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t \tag{9.2.10}$$

This solution shows that the mass oscillates about the rest position $x = 0$ with a frequency of $\omega_n = \sqrt{k/m}$ radians per unit time. The period of the oscillation is $2\pi/\omega_n$.



(a)

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

(b)

$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{L}}$$

(c)

$$LC\frac{d^2v}{dt^2} + v = v_s$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Figure 9.2.2 Examples of undamped systems.

Table 9.2.1 Solution forms for the free response of a second-order model.

Equation	Solution form
$\ddot{x} + a\dot{x} + bx = 0 \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	**1.** $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$
2. $(a^2 = 4b)$ repeated, real roots: s_1, s_1	**2.** $x(t) = (C_1 + tC_2)e^{s_1 t}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$ $\omega = \sqrt{b}$	**3.** $x(t) = C_1 \sin \omega t + C_2 \cos \omega t$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = r \pm j\omega$ $r = -a/2, \omega = \sqrt{4b - a^2}/2$	**4.** $x(t) = e^{rt}(C_1 \sin \omega t + C_2 \cos \omega t)$

The frequency of oscillation ω_n is called the *natural frequency*. The natural frequency is greater for stiffer springs (larger k values). The amplitude of the oscillation depends on the initial conditions $x(0)$ and $\dot{x}(0)$.

Damping is present in the model $m\ddot{x} + c\dot{x} + kx = f(t)$ if $c \neq 0$. If so, Cases 1, 2, and 4 in Table 9.2.1 summarize the free response forms. See Example 4.4.4 for specific examples of the damped free response. Figure 9.2.3 shows the free response for four values of c, with $m = 1$, $k = 16$, $x(0) = 1$, and $\dot{x}(0) = 0$. For no damping, the system is neutrally stable and the mass oscillates with a constant amplitude and a radian frequency of $\sqrt{k/m} = 4$, which is the natural frequency ω_n. As the damping is increased slightly to $c = 4$, the system becomes stable and the mass still oscillates but with a smaller radian frequency ($\sqrt{12} = 3.464$). The oscillations die out eventually as the mass returns to rest at the equilibrium position $x = 0$. As the damping is increased further to $c = 8$, the mass no longer oscillates because the damping force is large enough to limit the velocity and thus the momentum of the mass to a value that prevents the mass from overshooting the equilibrium position. For a larger value of c ($c = 10$), the mass takes longer to return to equilibrium because the damping force greatly slows down the mass.

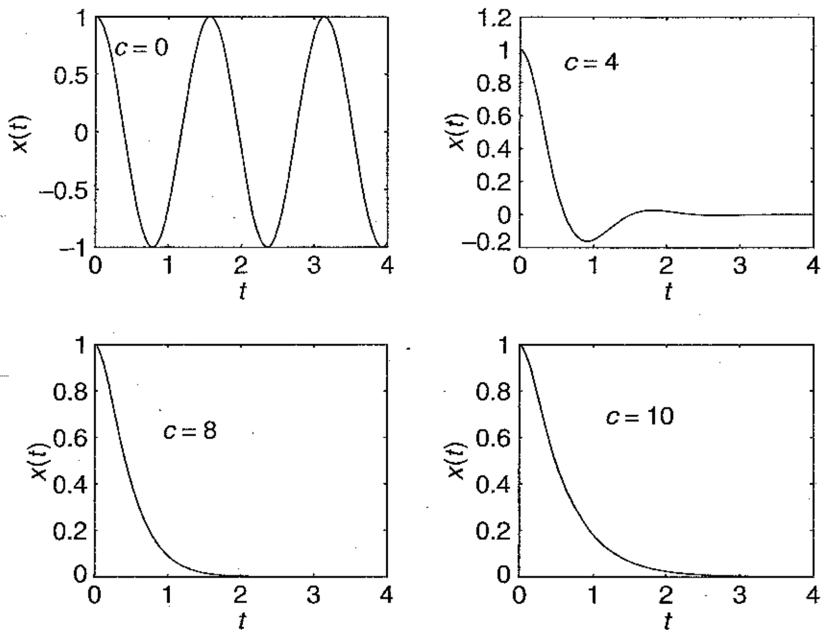


Figure 9.2.3 Responses for four values of c.

EFFECT OF ROOT LOCATION

Figure 9.2.4 shows how the location of the characteristic roots in the complex plane affects the free response. The real part of the root is plotted on the horizontal axis, and the imaginary part is plotted on the vertical axis. Because the roots are conjugate, we show only the upper root. Using the results we have found in earlier chapters, we see that

1. Unstable behavior occurs when the root lies to the right of the imaginary axis.
2. Neutrally stable behavior occurs when the root lies on the imaginary axis.
3. The response oscillates only when the root has a nonzero imaginary part.
4. The greater the imaginary part, the higher the frequency of the oscillation.
5. The farther to the left the root lies, the faster the response decays.

The characteristic equation of the model (9.2.1) is

$$ms^2 + cs + k = 0 \tag{9.2.11}$$

Its roots are

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = r \pm j\omega \tag{9.2.12}$$

where r and ω denote the real and imaginary parts of the roots.

THE DAMPING RATIO

A second-order system's free response for the stable case can be conveniently characterized by the *damping ratio* ζ (sometimes called the *damping factor*). For the characteristic equation (9.2.11), the damping ratio is defined as

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{9.2.13}$$

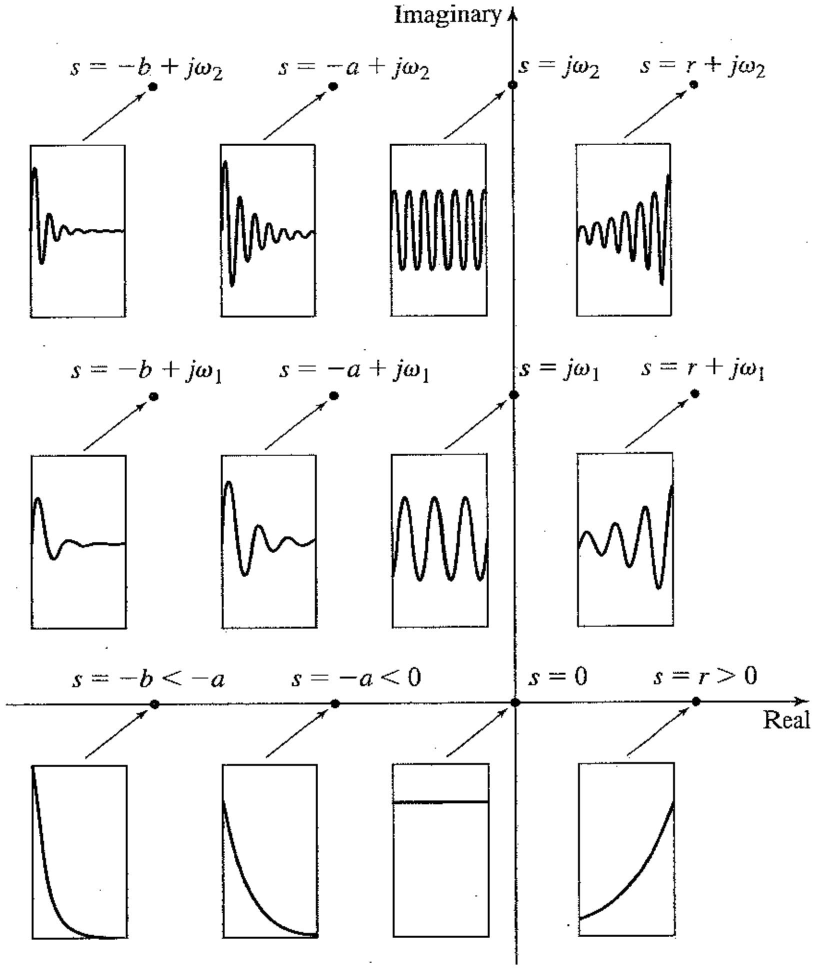


Figure 9.2.4 Effect of root location on the free response.

This definition is not arbitrary but is based on the way the roots change from real to complex as the value of c is changed; that is, from (9.2.12) we see that three cases can occur:

1. *The Critically Damped Case:* Repeated roots occur if $c^2 - 4mk = 0$; that is, if $c = 2\sqrt{mk}$. This value of the damping constant is the *critical damping constant* c_c, and when c has this value the system is said to be *critically damped.*
2. *The Overdamped Case:* If $c > c_c = 2\sqrt{mk}$, two real distinct roots exist, and the system is *overdamped.*
3. *The Underdamped Case:* If $c < c_c = 2\sqrt{mk}$, complex roots occur, and the system is *underdamped.*

The damping ratio is thus seen to be the ratio of the actual damping constant c to the critical value c_c. Note that

1. For a critically damped system, $\zeta = 1$.
2. Exponential behavior occurs if $\zeta > 1$ (the overdamped case).
3. Oscillations exist when $\zeta < 1$ (the underdamped case).

For an unstable system the damping ratio is meaningless and therefore not defined. For example, the equation $3s^2 - 5s + 4 = 0$ is unstable, and we do not compute its damping ratio, which would be negative if you used (9.2.13).

The damping ratio can be used as a quick check for oscillatory behavior. For example, the model whose characteristic equation is $s^2 + 5ds + 4d^2 = 0$ is stable if $d > 0$ and it has the following damping ratio.

$$\zeta = \frac{5d}{2\sqrt{4d^2}} = \frac{5}{4} > 1$$

Because $\zeta > 1$, no oscillations can occur in the system's free response regardless of the value of d and regardless of the initial conditions.

The motor transfer function (9.2.7) is repeated here.

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_T}{L_a I s^2 + (R_a I + c L_a)s + c R_a + K_b K_T}$$

The denominator of this model has the standard form $ms^2 + cs + k$, and thus the damping ratio of the motor model is, from (9.2.13),

$$\zeta = \frac{R_a I + c L_a}{2\sqrt{L_a I (R_a c + K_b K_T)}}$$

Even if the damping constant c is zero, the damping ratio is still nonzero because of the term $R_a I$.

NATURAL AND DAMPED FREQUENCIES OF OSCILLATION

The roots of (9.2.11) are purely imaginary when there is no damping. The imaginary part and the frequency of oscillation for this case is the *undamped natural frequency* $\omega_n = \sqrt{k/m}$.

We can write the characteristic equation in terms of the parameters ζ and ω_n. First divide (9.2.11) by m and use the fact that $\omega_n^2 = k/m$ and that

$$2\zeta\omega_n = 2\left(\frac{c}{2\sqrt{mk}}\right)\left(\sqrt{\frac{k}{m}}\right) = \frac{c}{m}$$

The characteristic equation becomes

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{9.2.14}$$

and the roots are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{9.2.15}$$

The frequency of oscillation is $\omega_n\sqrt{1-\zeta^2}$ and is called the *damped natural frequency,* or simply the damped frequency, to distinguish it from the undamped natural frequency ω_n. We will denote the damped frequency by ω_d.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{9.2.16}$$

The frequencies ω_n and ω_d have meaning only for the underdamped case ($\zeta < 1$). For this case (9.2.16) shows that $\omega_d < \omega_n$. Thus the damped frequency is always less than the undamped frequency.

Table 9.2.2 Free response of $m\ddot{x} + c\dot{x} + kx = f(t)$ for the stable case.

Characteristic roots	$s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
Undamped natural frequency	$\omega_n = \sqrt{\dfrac{k}{m}}$
Damping ratio	$\zeta = \dfrac{c}{2\sqrt{km}}$
Damped natural frequency	$\omega_d = \omega_n\sqrt{1-\zeta^2}$
Overdamped case ($\zeta > 1$)	Distinct, real roots: $s = -r_1, s = -r_2 \quad (r_1 \neq r_2)$ $x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t}$ $A_1 = \dfrac{\dot{x}(0) + r_2 x(0)}{r_2 - r_1}$ $A_2 = \dfrac{-r_1 x(0) - \dot{x}(0)}{r_2 - r_1} = x(0) - A_1$
Critically damped case ($\zeta = 1$)	Repeated roots: $s = -r_1, s = -r_1$ $x(t) = (A_1 + A_2 t)\, e^{-r_1 t}$ $A_1 = x(0)$ $A_2 = \dot{x}(0) + r_1 x(0)$
Underdamped case ($0 \leq \zeta < 1$)	Complex conjugate roots: $s = -a \pm jb, b > 0$ $x(t) = De^{-at}\sin(bt + \phi)$ $D = +\dfrac{1}{b}\sqrt{[bx(0)]^2 + [\dot{x}(0) + ax(0)]^2}$ $\sin\phi = \dfrac{x(0)}{D} \quad \cos\phi = \dfrac{\dot{x}(0) + ax(0)}{bD}$
Alternative form for $0 \leq \zeta < 1$	$x(t) = e^{-\zeta\omega_n t}[A\sin\omega_d t + x(0)\cos\omega_d t]$ $A = \dfrac{\zeta}{\sqrt{1-\zeta^2}}x(0) + \dfrac{1}{\omega_d}\dot{x}(0)$

TIME CONSTANT

Comparison of (9.2.12) with (9.2.15) shows that $r = -\zeta\omega_n$ and $\omega = \omega_n\sqrt{1-\zeta^2}$. Because the time constant τ is $-1/r$, we have

$$\tau = \frac{1}{\zeta\omega_n} \tag{9.2.17}$$

Remember that this formula applies only if $\zeta \leq 1$ (otherwise, $\sqrt{1-\zeta^2}$ is imaginary).

Table 9.2.2 summarizes the free response of the stable, linear, second-order model in terms of the parameters ζ, ω_n, and ω_d. Table 9.2.3 summarizes the formulas for these parameters.

GRAPHICAL INTERPRETATION

The preceding relations can be represented graphically by plotting the location of the roots (9.2.15) in the complex plane (Figure 9.2.5). The parameters ζ, ω_n, ω_d, and τ are normally used to describe stable systems only, and so we will assume for now that all the roots lie to the left of the imaginary axis (in the "left half-plane").

Table 9.2.3 Response parameters for second-order systems.

Model	$m\ddot{x} + c\dot{x} + kx = f(t)$ m, c, k constant
Characteristic Equation	$ms^2 + cs + k = 0$
1. Roots	$s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
2. Stability Property	Stable if and only if both roots have negative real parts. This occurs if and only if m, c, and k have the same sign.
3. Alternative forms for underdamped systems	
Characteristic Equation:	$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
Roots:	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
4. Damping ratio or damping factor	$\zeta = \dfrac{c}{2\sqrt{mk}}$
5. Undamped natural frequency	$\omega_n = \sqrt{\dfrac{k}{m}}$
6. Damped natural frequency	$\omega_d = \omega_n\sqrt{1-\zeta^2}$
7. Time constant	$\tau = 2m/c = 1/\zeta\omega_n \quad \text{if } \zeta \le 1$

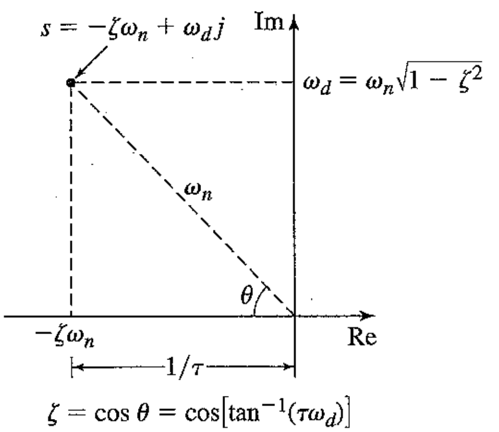


Figure 9.2.5 Graphical interpretation of the parameters ζ, τ, ω_n, and ω_d.

The lengths of two sides of the right triangle shown in Figure 9.2.5 are $\zeta\omega_n$ and $\omega_n\sqrt{1-\zeta^2}$. Thus the hypotenuse is of length ω_n. It makes an angle θ with the *negative* real axis, and

$$\cos\theta = \zeta \tag{9.2.18}$$

Therefore all roots lying on the circumference of a given circle centered on the origin are associated with the same undamped natural frequency ω_n. From (9.2.18) we see that all roots lying on the same line passing through the origin are associated with the same damping ratio. The limiting values of ζ correspond to the imaginary axis ($\zeta = 0$) and the negative real axis ($\zeta = 1$). Roots lying on a given line parallel to the real axis all give the same damped natural frequency. All roots lying on a line parallel to the imaginary axis have the same time constant. The farther to the left this line is, the smaller the time constant.

9.3 DESCRIPTION AND SPECIFICATION OF STEP RESPONSE

We can express the free response and the step response of the underdamped second-order model $m\ddot{x} + c\dot{x} + kx = f(t)$ in terms of the parameters ζ and ω_n as follows. The form of the free response is

$$x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{9.3.1}$$

where B and ϕ depend on the initial conditions $x(0)$ and $\dot{x}(0)$. The unit step response for zero initial conditions is

$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + 1\right] \tag{9.3.2}$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi \tag{9.3.3}$$

Because $\zeta > 0$, ϕ lies in the third quadrant. Table 9.3.1 gives the step response in terms of ζ and ω_n for the underdamped, critically damped, and overdamped cases.

The free response for $\dot{x}(0) = 0$ and the step response for $x(0) = \dot{x}(0) = 0$ of the second-order model $m\ddot{x} + c\dot{x} + kx = f(t)$ are illustrated in Figures 9.3.1 and 9.3.2 for several values of the damping ratio ζ. Note that the response axis has been normalized by k and the time axis has been normalized by $\omega_n = \sqrt{k/m}$. Thus a variation of ζ should be interpreted as a variation of c, and not of k or m. When $\zeta > 1$, the response is

Table 9.3.1 Unit step response of a stable second-order model.

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. Overdamped case ($\zeta > 1$): distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2}e^{-r_1 t} - \frac{r_1}{r_1 - r_2}e^{-r_2 t} + 1\right)$$

2. Critically damped case ($\zeta = 1$): repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t)e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1)e^{-r_1 t} + 1]$$

3. Underdamped case ($0 \le \zeta < 1$): complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$x(t) = Be^{-t/\tau}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right) + 1\right]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi \quad \text{(third quadrant)}$$

Time constant: $\tau = 1/\zeta\omega_n$

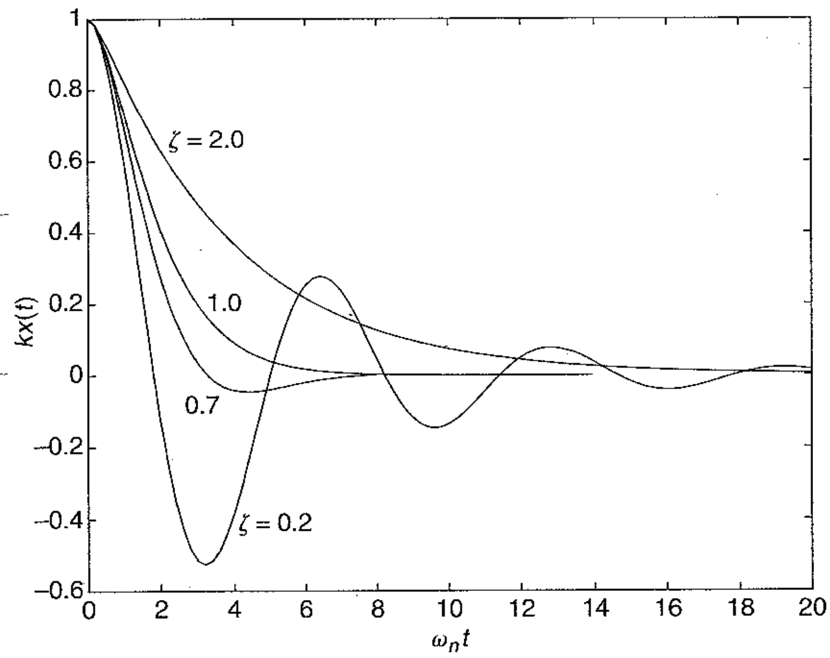


Figure 9.3.1 Free response of second-order systems for various values of ζ.

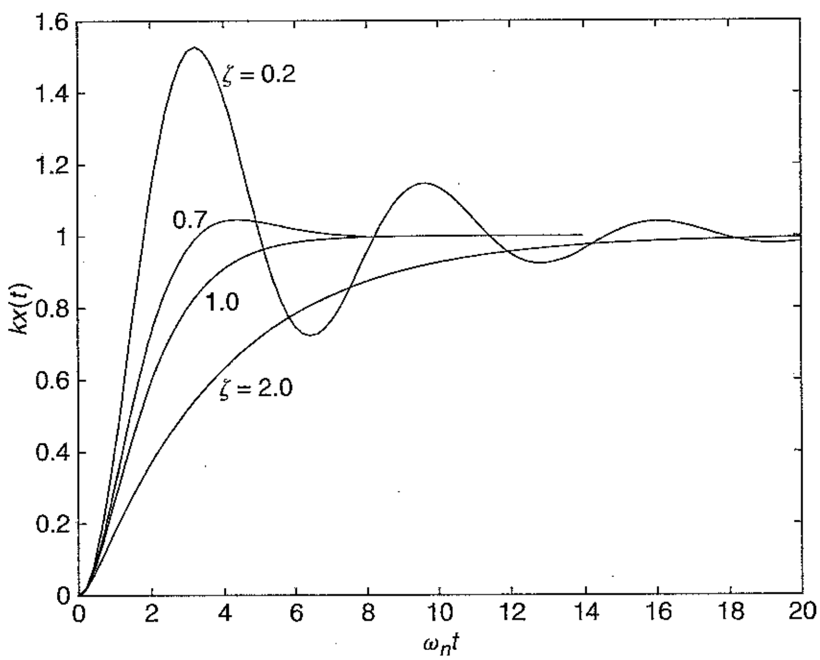


Figure 9.3.2 Step response of second-order systems for various values of ζ.

sluggish and does not overshoot the steady-state value. As ζ is decreased, the speed of response increases. The critically damped case $\zeta = 1$ is the case in which the steady-state value is reached most quickly but without oscillation. As ζ is decreased below 1, the response overshoots and oscillates about the final value. The smaller ζ is, the larger the overshoot, and the longer it takes for the oscillations to die out. There are design applications in which we wish the response to be near its final value as quickly as possible, with some oscillation tolerated. This corresponds to a value of ζ slightly less than 1. The value $\zeta = 0.707$ is a common choice for such applications. As ζ is decreased to zero (the undamped case), the oscillations never die out.

DESCRIPTION OF STEP RESPONSE

Suppose you obtained the response plot shown in Figure 9.3.3 either from a measured response or from a computer simulation. Suppose also that you want to describe the plot to someone over the phone (assuming you cannot send them the plot!). You would say that $x(t)$ starts at 0 and rises to the steady-state value of $x = 100$ mm. It oscillates briefly around the steady-state value (there are about two oscillations with a period of about 6.6 s). The first oscillation has the largest peak, which is 37 mm and occurs at $t = 3.3$ s. You might also note that the response reaches 50% of the steady-state value in 1.2 s, and it first reaches 100% of the steady-state value in 2 s. Note that, depending on the resolution of the plot, you might not be able to determine the time for the response to reach the steady state, so this time estimate might be subject to great error.

The parameters we have just used to describe the plot not only are the standard ways of describing step response, but also are the standard ways of specifying desired performance. These transient-response specifications are illustrated in Figure 9.3.4. Note that the response need not be that of a second-order system. The *maximum or peak overshoot* M_p is the maximum deviation of the output x above its steady-state value x_{ss}. It is sometimes expressed as a percentage of the final value and denoted $M_{\%}$. Because the maximum overshoot increases with decreasing ζ, it is sometimes used as an indicator of the relative stability of the system. The *peak time* t_p is the time at which the maximum overshoot occurs. The *settling time* t_s is the time required for the oscillations to stay within some specified small percentage of the final value. The most common values used are 2% and 5%. If the final value of the response differs from some desired value, a steady-state error exists.

The *rise time* t_r can be defined as the time required for the output to rise from 0% to 100% of its final value. However, no agreement exists on this definition. Sometimes, the rise time is taken to be the time required for the response to rise from 10% to 90% of the final value. Finally, the *delay time* t_d is the time required for the response to reach 50% of its final value.

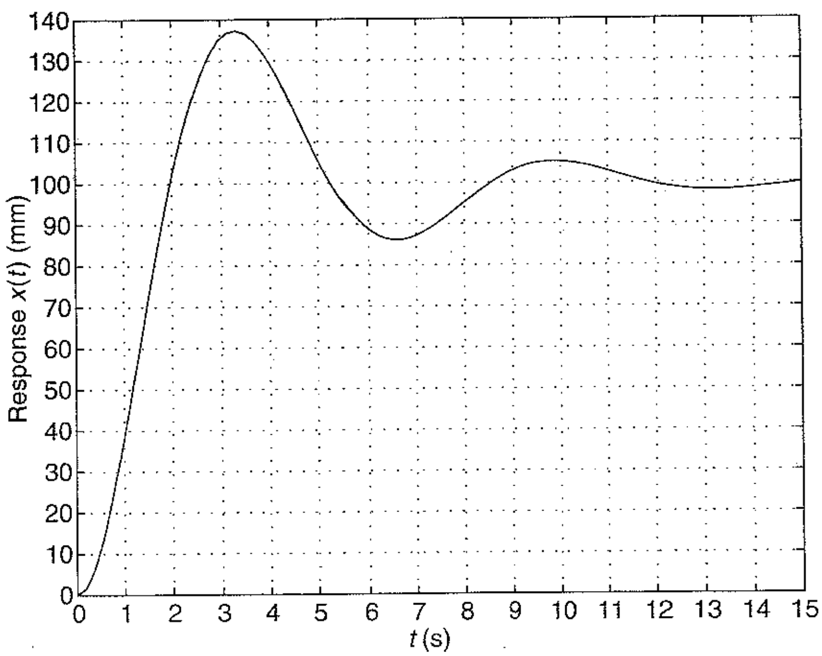


Figure 9.3.3 Underdamped step response.

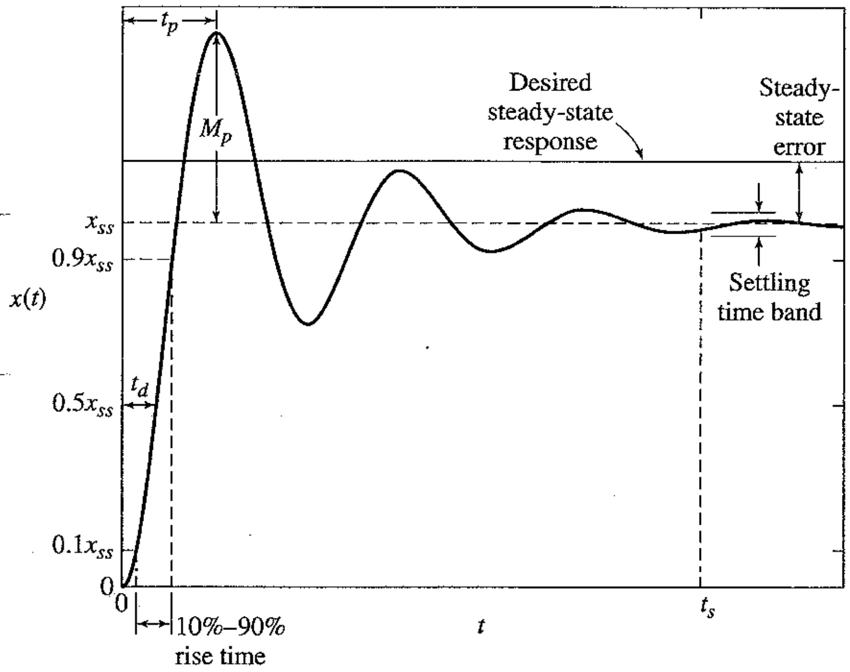


Figure 9.3.4 Transient-response specifications.

Except for the settling time, these parameters are relatively easy to obtain from an experimentally determined step-response plot. They can also be determined in analytical form for a second-order model, whose underdamped step response is given by (9.3.2) and (9.3.3).

MAXIMUM OVERSHOOT

Setting the derivative of (9.3.2) equal to zero gives expressions for both the maximum overshoot and the peak time t_p. After some trigonometric manipulation, the result is

$$\frac{dx}{dt} = \frac{1}{k}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\,\omega_n\sqrt{1-\zeta^2}t\right) = 0$$

For $t < \infty$, this gives

$$\omega_n\sqrt{1-\zeta^2}t = n\pi \quad n = 0, 1, 2, \ldots$$

The times at which extreme values of the oscillations occur are thus

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{9.3.4}$$

The odd values of n give the times of overshoots, and the even values correspond to the times of undershoots. The maximum overshoot occurs when $n = 1$. Thus,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{9.3.5}$$

The magnitudes $|x_n|$ of the overshoots and undershoots are found by substituting (9.3.4) into (9.3.2). After some manipulation, the result is

$$|x_n| = \frac{1}{k}\left[1 + (-1)^{n-1}e^{-n\pi\zeta/\sqrt{1-\zeta^2}}\right] \tag{9.3.6}$$

The largest value $|x_n|$ occurs when $n = 1$. Thus the maximum overshoot M_p is found when $n = 1$. It is

$$M_p = x_{\max} - x_{ss} = \frac{1}{k} e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{9.3.7}$$

The preceding expressions show that the maximum overshoot and the peak time are functions of only the damping ratio ζ for a second-order system. The *percent* maximum overshoot $M_{\%}$ is

$$M_{\%} = \frac{x_{\max} - x_{ss}}{x_{ss}} 100 = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{9.3.8}$$

Frequently we need to compute the damping ratio from a measured value of the maximum overshoot. In this case, we can solve (9.3.8) for ζ as follows. Let $R = \ln 100/M_{\%}$. Then (9.3.8) gives

$$\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} \qquad R = \ln \frac{100}{M_{\%}} \tag{9.3.9}$$

RISE TIME

To obtain the expression for the 100% rise time t_r, set $x = x_{ss} = 1/k$ in (9.3.2) to obtain

$$e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \phi\right) = 0$$

This implies that for $t < \infty$,

$$\omega_n \sqrt{1-\zeta^2}\, t + \phi = n\pi \qquad n = 0, 1, 2, \ldots \tag{9.3.10}$$

For $t_r > 0$, $n = 2$, because ϕ is in the third quadrant. Thus,

$$t_r = \frac{2\pi - \phi}{\omega_n \sqrt{1-\zeta^2}} \tag{9.3.11}$$

where ϕ is given by (9.3.3). The rise time is inversely proportional to the natural frequency ω_n for a given value of ζ.

SETTLING TIME

To express the settling time in terms of the parameters ζ and ω_n, we can use the fact that the exponential term in the solution (9.3.2) provides the envelopes of the oscillations. The time constant of these envelopes is $1/\zeta\omega_n$, and thus the 2% settling time t_s is

$$t_s = \frac{4}{\zeta\omega_n} \tag{9.3.12}$$

DELAY TIME

An exact analytical expression for the delay time is difficult to obtain, but we can obtain an approximate expression as follows. Set $x = 0.5x_{ss} = 0.5/k$ in (9.3.2) to obtain

$$e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \phi\right) = -0.5\sqrt{1-\zeta^2} \tag{9.3.13}$$

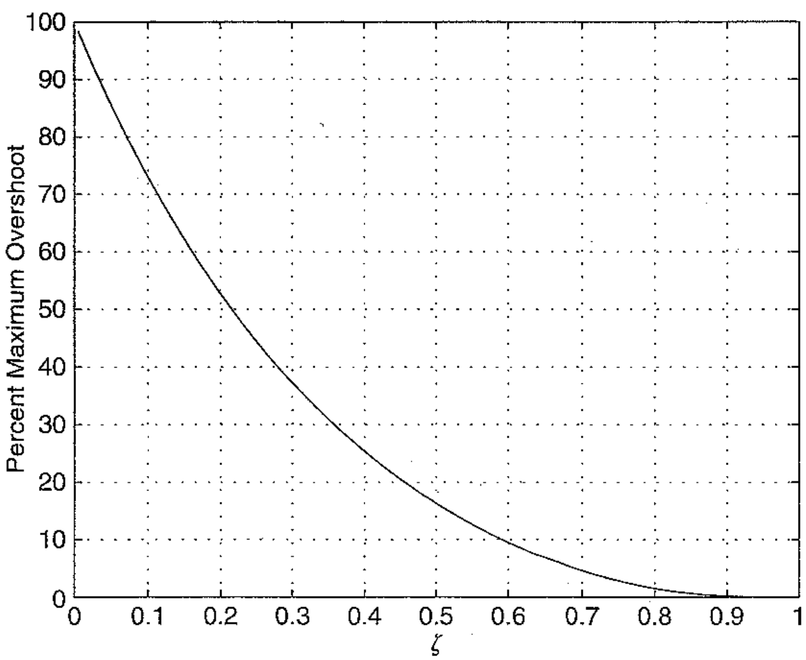
where ϕ is given by (9.3.3). For a given ζ and ω_n, t_d can be obtained by a numerical procedure, using the following straight-line approximation as a starting guess:

$$t_d \approx \frac{1 + 0.7\zeta}{\omega_n} \tag{9.3.14}$$

Table 9.3.2 summarizes these formulas. Figure 9.3.5 shows the plots of the maximum percent overshoot, the peak time, and the 100% rise time as functions of ζ. In Section 9.5 we will see how these formulas can be used with experimentally determined response plots to estimate the parameters m, c, and k. In Chapters 10, 11, and 12, we will use these specifications to describe the desired performance of control systems.

Table 9.3.2 Step response specifications for the underdamped model $m\ddot{x} + c\dot{x} + kx = f$.

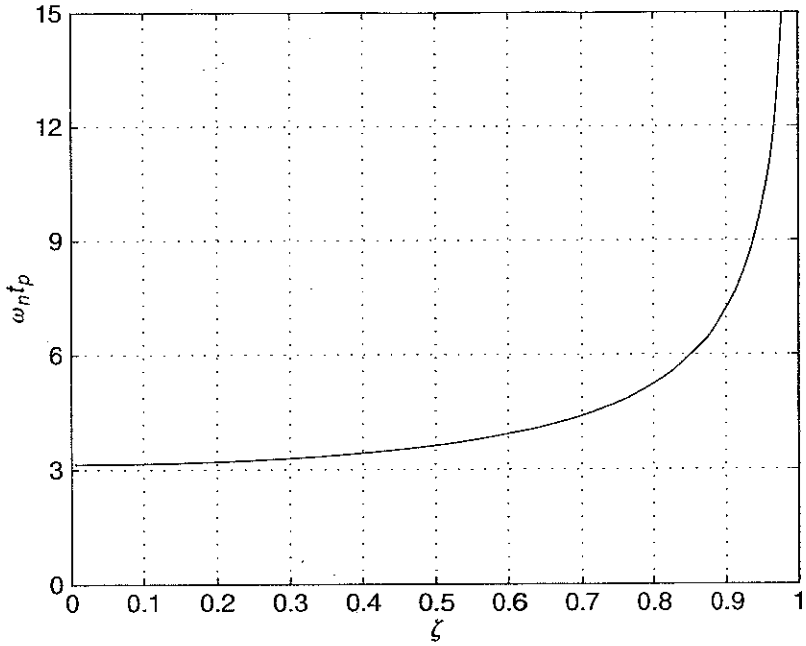
Maximum percent overshoot	$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
	$\zeta = \dfrac{R}{\sqrt{\pi^2 + R^2}}, \quad R = \ln \dfrac{100}{M_\%}$
Peak time	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Delay time	$t_d \approx \dfrac{1 + 0.7\zeta}{\omega_n}$
100% rise time	$t_r = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$
	$\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$



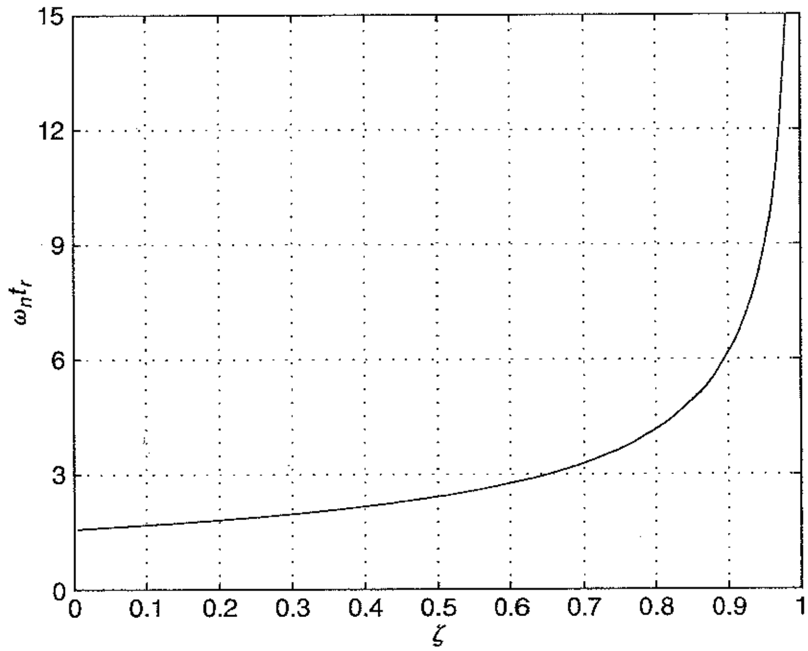
(a)

Figure 9.3.5 Percent maximum overshoot, peak time, and rise time as functions of the damping ratio ζ. *(Continued)*

Figure 9.3.5 *(Continued)*



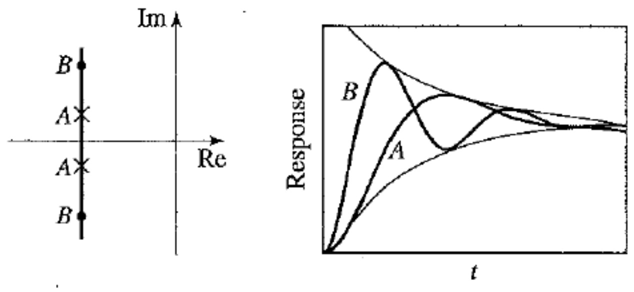
(b)



(c)

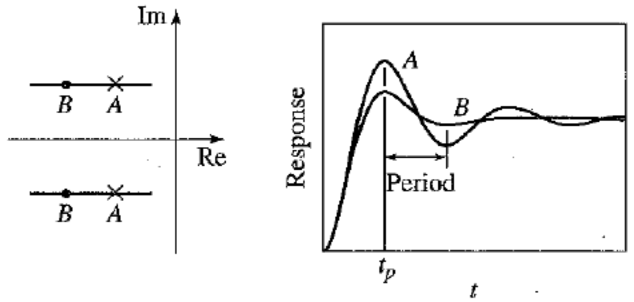
Figure 9.3.6 illustrates the effect of root location on decay rate, peak time, and overshoot. In part (a) of the figure, we see that roots lying on the same vertical line have the same decay rate because they have the same time constant. Part (b) shows that roots lying on the same horizontal line have the same oscillation frequency, period, and peak time. Part (c) of the figure shows that roots lying on the same radial line have the same damping ratio and therefore the same maximum percent overshoot $M_{\%}$. You can

Models A and B have the same real part, the same time constant, and the same decay time.



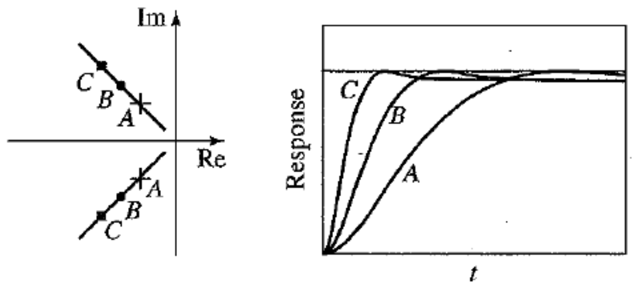
(a)

Models A and B have the same imaginary part, the same period, and the same peak time.



(b)

Models A, B, and C have the same damping ratio and the same overshoot.



(c)

Figure 9.3.6 Effect of root location on decay rate, peak time, and overshoot.

see this from the $M_{\%}$ formula, (9.3.8), which is a function of ζ only. Roots lying on the same horizontal line have the same damped frequency ω_d, and therefore have the same peak time t_p. This is true because $t_p = \pi/\omega_n\sqrt{1-\zeta^2}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, and therefore, $t_p = \pi/\omega_d$. Thus if ω_d is constant, so is t_p.