CHAPTER 8

Additional Problems

8.16 Provided that the channel bandwidth is not smaller than the reciprocal of the transmitted pulse duration $T$, the received pulse is recognizable at the channel output. With $T = 1\mu s$, a small enough value for the channel bandwidth is $(1/T) = 10^6$ Hz = 1 MHz.

8.17 For a low-pass filter of the Butterworth type, the squared magnitude response is defined by

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \quad (1)$$

At the edge of the passband, $\omega = \omega_p$, we have (by definition)

$$|H(j\omega_p)| = 1 - \varepsilon$$

We may therefore write

$$(1 - \varepsilon)^2 = \frac{1}{1 + (\omega_p/\omega_c)^{2N}} \quad (2)$$

Define

$$\varepsilon_0 = 1 - (1 - \varepsilon)^2 = 2\varepsilon - \varepsilon^2$$

Then solving Eq. (2) for $\omega_p$:

$$\omega_p = \left( \frac{\varepsilon_0}{1 - \varepsilon_0} \right) \omega_c$$

Next, by definition, at the edge of the stopband, $\omega = \omega_s$, we have

$$|H(j\omega_s)| = \delta$$

Hence

$$\delta^2 = \frac{1}{1 + (\omega_s/\omega_c)^{2N}} \quad (3)$$

Define

$$\delta_0 = \delta^2$$

Hence, solving Eq. (3) for $\omega_s$:

$$\omega_s = \left( \frac{1 - \delta_0}{\delta_0} \right) \omega_c$$
8.18 We start with the relation
\[ |H(j\omega)|_{j\omega=s}^2 = H(s)H(-s) \]
For a Butterworth low-pass filter of order 5, the 10 poles of \( H(s)H(-s) \) are uniformly distributed around the unit circle in the \( s \)-plane as shown in Fig. 1.

Let \( D(s)D(-s) \) denote the denominator polynomial of \( H(s)H(-s) \). Hence
\[
D(s)D(-s) = (s + 1)(s + \cos 144^\circ + j\sin 144^\circ)(s + \cos 144^\circ - j\sin 144^\circ) \\
\times (s + \cos 108^\circ + j\sin 108^\circ)(s + \cos 108^\circ - j\sin 108^\circ) \\
\times (s - 1)(s - \cos 36^\circ + j\sin 36^\circ)(s - \cos 36^\circ - j\sin 36^\circ) \\
\times (s - \cos 72^\circ + j\sin 72^\circ)(s - \cos 72^\circ - j\sin 72^\circ)
\]
Identifying the zeros of \( D(s)D(-s) \) in the left-half plane with \( D(s) \) and those in the right-half plane with \( D(-s) \), we may express \( D(s) \) as
\[
D(s) = (s + 1)(s + \cos 144^\circ + j\sin 144^\circ)(s + \cos 144^\circ - j\sin 144^\circ) \\
\times (s + \cos 108^\circ + j\sin 108^\circ)(s + \cos 108^\circ - j\sin 108^\circ) \\
= s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1
\]
Hence,
\[
H(s) = \frac{1}{D(s)} \\
= \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1}
\]

8.19 (a) For filter order \( N \) that is odd, the transfer function \( H(s) \) of the filter must have a real pole in the left-half plane. Let this pole be \( s = -a \) where \( a > 0 \). We may then write
\[
H(s) = \frac{1}{(s + a)D'(s)}
\]
where \( D'(s) \) is the remainder of the denominator polynomial. For a Butterworth low-pass filter of cutoff frequency \( \omega_c \), all the poles of \( H(s) \) lie on a circle of radius \( \omega_c \) in the left-half plane. Hence, we must have \( a = -\omega_c \).

(b) For a Butterworth low-pass filter of even order \( N \), all the poles of the transfer function \( H(s) \) are complex. They all lie on a circle of radius \( \omega_c \) in the left-half plane. Let \( s = -a - jb \), with \( a > 0 \) and \( b > 0 \), denote a complex pole of \( H(s) \). All the coefficients of \( H(s) \) are real. This condition can only be satisfied if we have a complex conjugate pole at \( s = -a + jb \). We may then express the contribution of this pair of poles as

\[
\frac{1}{(s + a + jb)(s + a - jb)} = \frac{1}{(s + a)^2 + b^2}
\]

whose coefficients are all real. We therefore conclude that for even filter order \( N \), all the poles of \( H(s) \) occur in complex-conjugate pairs.

\[8.20\] The transfer function of a Butterworth low-pass filter of order 5 is

\[
H(s) = \frac{1}{(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)}
\]

The low-pass to high-pass transformation is defined by

\[ s \rightarrow \frac{1}{s} \]

where it is assumed that the cutoff frequency of the high-pass filter is unity. Hence replacing \( s \) with \( 1/s \) in Eq. (1), we find that the transfer function of a Butterworth high-pass filter of order 5 is

\[
H(s) = \frac{1}{(\frac{1}{s} + 1)(\frac{1}{s} + 0.618s + 1)(\frac{1}{s} + 1.618s + 1)}
\]

\[
= \frac{s^5}{(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)}
\]

The magnitude response of this high-pass filter is plotted in Fig. 1.

![Figure 1](image-url)
8.21 We are given the transfer function

\[ H(s) = \frac{1}{(s + 1)(s^2 + 0.618s + 1)(s + 1.618s + 1)} \]  

(1)

To modify this low-pass filter so as to assume a cutoff frequency \( \omega_c \), we use the transformation

\[ s \rightarrow \frac{s}{\omega_c} \]

Hence, replacing \( s \) with \( s/\omega_c \) in Eq. (1), we obtain:

\[
H(s) = \frac{1}{\left(\frac{s}{\omega_c} + 1\right)\left(\frac{s^2}{\omega_c^2} + 0.618\frac{s}{\omega_c} + 1\right)\left(\frac{s^2}{\omega_c^2} + 1.618\frac{s}{\omega_c} + 1\right)}
\]

\[ = \frac{\omega_c^5}{(s + \omega_c)(s^2 + 0.618\omega_c s + \omega_c^2)(s^2 + 1.618\omega_c s + \omega_c^2)} \]

8.22 We are given the transfer function

\[ H(s) = \frac{1}{(s + 1)(s^2 + s + 1)} \]  

(1)

To transform this low-pass filter into a band-pass filter with bandwidth \( B \) centered on \( \omega_0 \), we use the following transformation:

\[ s \rightarrow \frac{s^2 + \omega_0^2}{B \ s} \]

With \( \omega_0 = 1 \) and \( B = 0.1 \), the transformation takes the value

\[ s \rightarrow \frac{s^2 + 1}{0.1s} \]  

(2)

Substituting Eq. (2) into (1):

\[
H(s) = \frac{1}{\left(\frac{s^2 + 1}{0.1s} + 1\right)\left(\left(\frac{s + 1}{0.1s}\right)^2 + \left(\frac{s + 1}{0.1s}\right) + 1\right)}
\]

\[ = \frac{0.001s^3}{(s^2 + 0.1s + 1)(s^4 + 2s^2 + 1 + 0.1s^3 + 0.1s + 0.01s^2)} \]
The magnitude response of this band-pass filter of the Butterworth type, obtained by putting \( s = j\omega \) in Eq. 3, is plotted in Fig. 1.

\[
\frac{0.001s^3}{(s^2 + 0.1s + 1)(s^4 + 0.1s^3 + 2.01s^2 + 0.1s + 1)}
\]

(3)

8.23 For the counterpart to the low-pass filter of order one in Fig. 8.14(a), we have

\[ v_1(t) \quad \mid \quad 1\Omega \quad \mid \quad IF \quad \mid \quad v_2(t) \]

Input \quad Filter \quad Output
For the counterpart to the low-pass filter of order three in Fig. 8.14(b), we have

![Filter Diagram]

Note that in both of these figures, the 1Ω resistance may be used to account for the source resistance. Note also that the load resistance is infinitely large.

8.24  (a) For Fig. 8.14(a), the capacitor has the value

\[ C = \frac{1}{(10^4)(2\pi \times 10^5)} \text{F} \]

\[ = \frac{10^3}{2\pi} \text{pF} \]

\[ = 159 \text{ pF} \]

(b) For Fig. 8.14(a), the two capacitors are 159 PF each. The inductor has the value

\[ L = \frac{(10^4)}{(2\pi \times 10^5)} \text{H} \]

\[ = \frac{100}{2\pi} \text{mH} \]

\[ = 15.9 \text{ mH} \]

8.25  Let the transfer function of the FIR filter be defined by

\[ H(z) = (1 - z^{-1})A(z) \]

(1)

where \( A(z) \) is an arbitrary polynomial in \( z^{-1} \). The \( H(z) \) of Eq. (1) has a zero at \( z = 1 \) as prescribed. Let the sequence \( a[n] \) denote the inverse \( z \)-transform of \( A(z) \). Expanding Eq. (1):

\[ H(z) = A(z) - z^{-1}A(z) \]

(2)
which may be represented by the block diagram:

![Block diagram](image)

From Eq. (2) we readily find that the impulse response of the filter must satisfy the condition

\[ h[n] = a[n] - a[n - 1] \]

where \( a[n] \) is the inverse \( z \)-transform of an arbitrary polynomial \( A(z) \).

8.26 Let the transfer function of the filter be defined by

\[ H'(e^{j\Omega}) = \sum_{n=M/2}^{M/2} h_d[n] e^{-jn\Omega} \]

Replacing \( n \) with \( n - M/2 \) so as to make the filter causal, we may thus write

\[
H'(e^{j\Omega}) = \sum_{n=0}^{M} h_d\left[n - \frac{M}{2}\right] e^{-j\left(n - \frac{M}{2}\right)\Omega} \\
= e^{j\Omega/2} \sum_{n=0}^{M} h_d\left[n - \frac{M}{2}\right] e^{-jn\Omega} \\
= e^{j\Omega/2} \sum_{n=0}^{M} h_d[n] e^{-jn\Omega} \\
= e^{j\Omega/2} \sum_{n=0}^{N} h[n] e^{-jn\Omega} \\
= e^{j\Omega/2} H(e^{j\Omega})
\]

We are given that

\[ h[n] = h_d[n] \quad \text{for} \quad -\frac{M}{2} \leq n \leq \frac{M}{2} \]

This condition is equivalent to

\[ h_d\left[n - \frac{M}{2}\right] = h[n] \quad \text{for} \quad 0 \leq n \leq M \]

Hence, we may rewrite Eq. (1) in the form

\[ H'(e^{j\Omega}) = e^{jM\Omega/2} \sum_{n=0}^{N} h[n] e^{-jn\Omega} \]

Equivalently we have

\[ H(e^{j\Omega}) = e^{-jM\Omega/2} H'(e^{j\Omega}) \]

which is the desired result.
According to Eqs. (8.64) and (8.65), the magnitude \( r = |z| \) and phase \( \theta = \arg(z) \) are defined by

\[
r = \left( \frac{(1 + \sigma)^2 + \omega^2}{(1 - \sigma)^2 + \omega^2} \right)^{1/2} \tag{1}
\]

\[
\theta = \tan^{-1}\left( \frac{\omega}{1 + \sigma} \right) - \tan^{-1}\left( \frac{\omega}{1 - \sigma} \right) \tag{2}
\]

These two relations are based on the transformation

\[
z = re^{j\theta} = \frac{1 + s}{1 - s}, \quad s = \sigma + j\omega
\]

For the more general case of a sampling rate \( 1/T_s \) for which we have

\[
s = \frac{1}{T_s} \frac{z - 1}{z + 1}
\]

or

\[
z = \frac{1 + \frac{T_s}{2} s}{1 - \frac{T_s}{2} s}
\]

we may rewrite Eqs. (1) and (2) by replacing \( \omega \) with \( \frac{T_s}{2} \omega \) and \( \sigma \) with \( \frac{T_s}{2} \sigma \), obtaining

\[
r = \left( \frac{\left( \frac{2}{T_s} + \sigma \right)^2 + \omega^2}{\left( \frac{2}{T_s} - \sigma \right)^2 + \omega^2} \right)^{1/2}
\]

\[
\theta = \tan^{-1}\left( \frac{\omega}{\frac{2}{T_s} + \sigma} \right) - \tan^{-1}\left( \frac{\omega}{\frac{2}{T_s} - \sigma} \right)
\]

(a) From Section 1.10, we recall the input-output relation

\[y[n] = x[n] + \rho y[n - 1]\]

Taking \( z \)-transforms:

\[Y(z) = X(z) + \rho z^{-1} Y(z)\]

The transfer function of the filter is therefore
For \( \rho = 1 \), we have

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \rho z^{-1}}
\]  \hspace{1cm} (1)

For \( \rho = 1 \), we have

\[
H(z) = \frac{1}{1 - z^{-1}}
\]  \hspace{1cm} (2)

For \( z = j\Omega \), the frequency response of the filter is defined by

\[
H(e^{j\Omega}) = \frac{1}{1 - e^{-j\Omega}}
\]

with

\[
|H(e^{j\Omega})| = \frac{1}{|1 - e^{-j\Omega}|} = \frac{1}{[(1 - \cos\Omega)^2 + \sin^2\Omega]^{1/2}} = \frac{1}{(2 - 2\cos\Omega)^{1/2}}
\]

which is plotted in Fig. 1 for \( 0 \leq \Omega \leq \pi \). From this figure we see that the filter defined in Eq. (2) does not deviate from the ideal integrator by more than 1% for \( 0 \leq \Omega \leq 0.49 \).
(b) For $\rho = 0.99$, the use of Eq. (1) yields

$$H(z) = \frac{1}{1 - 0.99z^{-1}}$$

(3)

for which the frequency response is defined by

$$H(e^{j\Omega}) = \frac{1}{1 - 0.99e^{-j\Omega}}$$

That is,

$$|H(e^{j\Omega})| = \frac{1}{|1 - 0.99e^{-j\Omega}|} = \frac{1}{[(1 - 0.99\cos\Omega)^2 + (0.99\sin\Omega)^2]^{1/2}} = \frac{1}{(1.98 - 1.98\cos\Omega)^{1/2}}$$

which is plotted in Fig. 2. From this second figure we see that the usable range of the filter of Eq. (3) as an integrator is reduced to 0.35.

Figure 2

8.29 The transfer function of the digital IIR filter is

$$H(z) = \frac{0.0181(z + 1)^3}{(z - 0.50953)(z^2 - 1.2505z + 0.39812)}$$

Expanding the numerator and denominator polynomials of $H(z)$ in ascending powers of
\[ z^{-1}, \text{ we may write} \]
\[
H(z) = \frac{0.0181(1 + 3z^{-1} + 3z^{-2} + z^{-3})}{1 - 1.7564z^{-1} + 1.0308z^{-2} - 0.2014z^{-3}}
\]

Hence, the filter may be implemented in direct form II using the following configuration:

8.30 (a) The received signal, ignoring channel noise, is given by
\[
y(t) = x(t) + 0.1x(t - 10) + 0.2x(t - 15)
\]
where \( x(t) \) is the transmitted signal, and time \( t \) is measured in microseconds. Suppose \( y(t) \) is sampled uniformly with a sampling period of 5 \( \mu s \), yielding
\[
y[n] = x[n] + 0.1x[n - 2] + 0.2x[n - 3]
\]
(1)

(b) Taking the \( z \)-transforms of Eq. (1):
\[
Y(z) = X(z) + 0.1z^{-2}X(z) + 0.2z^{-3}X(z)
\]
which, in turn, yields the transfer function for the channel:
\[
H(z) = \frac{Y(z)}{X(z)} = 1 + 0.1z^{-2} + 0.2z^{-3}
\]
The corresponding equalizer is defined by the transfer function
\[
H_{eq}(z) = \frac{1}{H(z)} = \frac{1}{1 + 0.1z^{-2} + 0.2z^{-3}}
\]
which is realized by the IIR filter:
For $H_{eq}(z)$ to be stable, all of its three poles or, equivalently, all three roots of the cubic equation
\[ 1 + 0.1z^{-2} + 0.2z^{-3} = 0, \text{ or equivalently,} \]
\[ z^3 + 0.1z + 0.2 = 0 \]
must lie inside the unit circle in the $z$-plane. Using MATLAB, we find the roots are
\[ z = 0.2640 + j0.5560 \]
\[ z = 0.2640 - j0.5560 \]
\[ z = -0.5280 \]
which confirm stability of the IIR equalizer.

[Note: The stability of $H_{eq}(z)$ may also be explored using an indirect approach, namely, the Routh-Hurwitz criterion which avoids having to compute the roots. First, we use the bilinear transformation
\[ z = \frac{1+s}{1-s} \]
and then construct the Routh array as described in Section 9.12. The stability of $H_{eq}(z)$ is confirmed by examining the coefficients of the first column of the Routh array. The fact that all these coefficients are found to be positive assures the stability of $H_{eq}(z)$.
]

Realizing the equalizer by means of an FIR structure, we have
\[ H_{eq}(z) = (1 + 0.1z^{-2} + 0.2z^{-3})^{-1} \]
\[ = 1 - (0.1z^{-2} + 0.2z^{-3}) + (0.1z^{-2} + 0.2z^{-3})^2 - (0.1z^{-2} + 0.2z^{-3})^3 + \ldots \]
\[ = 1 - (0.1z^{-2} + 0.2z^{-3}) + (0.01z^{-4} + 0.04z^{-5} + 0.04z^{-6}) \]
\[ - (0.001z^{-6} + 0.006z^{-7} + 0.012z^{-8} + 0.008z^{-9} + \ldots) \]

Ignoring coefficients smaller than 1% as specified, we have the approximate result:
\[ H_{eq}(z) \approx 1 - 0.1z^{-2} - 0.2z^{-3} + 0.01z^{-4} + 0.04z^{-5} + 0.04z^{-6} - 0.012z^{-8} \]
which is realized using the following FIR structure:
The integrator output is
\[ y(t) = \int_{t-T_0}^{t} x(\tau)d\tau \tag{1} \]

Let \( x(t) \leftrightarrow X(j\omega) \). We may therefore reformulate the expression for \( y(t) \) as
\[
y(t) = \int_{t-T_0}^{t} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega\tau}d\omega \right) d\tau
\]

Interchanging the order of integration:
\[
y(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) \left( \int_{t-T_0}^{t} e^{j\omega\tau}d\tau \right) d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \left( \frac{T_0}{2\pi} \text{sinc} \left( \frac{\omega T_0}{2\pi} \right) e^{j\omega(t-T_0)/2} \right) d\omega \tag{2}
\]

(a) Invoking the formula for the inverse Fourier transform, we immediately deduce from Eq. (2) that the Fourier transform of the integrator output \( y(t) \) is given by
\[
Y(j\omega) = \frac{T_0}{2\pi} \text{sinc} \left( \frac{\omega T_0}{2\pi} \right) e^{-j\omega T_0/2} \tag{3}
\]
Examining this formula, we also readily see that \( y(t) \) can be equivalently obtained by passing the input signal \( x(t) \) through a filter whose frequency response is defined by
\[
H(j\omega) = \frac{T_0}{2\pi} \text{sinc} \left( \frac{\omega T_0}{2\pi} \right) e^{-j\omega T_0/2}
\]
The magnitude response of the filter is depicted in Fig. 1:
(b) Figure 1 also includes the magnitude response of an “approximating” ideal low-pass filter. This latter filter has a cutoff frequency $\omega_0 = 2\pi/T_0$ and passband gain of $T_0/2\pi$. Moreover, the filter has a constant delay of $T_0/2$. The response of this ideal filter to a step function applied at time $t = 0$ is given by

$$y'(t) = \frac{T_0}{\pi} \int_{-\infty}^{\infty} \frac{2\pi(t - T_0/2)}{\lambda} \sin \frac{\lambda}{T_0} d\lambda$$

At $t = T_0$, we therefore have

$$y'(T_0) = \frac{T_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\lambda}{T_0}}{\lambda} d\lambda$$

$$= \frac{T_0}{\pi} \left( \int_{-\infty}^{0} \frac{\sin \frac{\lambda}{T_0}}{\lambda} d\lambda + \int_{0}^{\infty} \frac{\sin \frac{\lambda}{T_0}}{\lambda} d\lambda \right)$$

$$= \frac{T_0}{\pi} (S_j(\infty) + S_j(\pi))$$

$$= 1.09 T_0$$

From Eq. (1) we find that the ideal integrator output at time $t = T_0$ in response to the step function $x(t) = u(t)$ is given by

$$y(T_0) = \int_{0}^{T_0} u(\tau) d\tau$$

It follows therefore that the output of the “approximating” ideal low-pass filter exceeds the output of the ideal integrator by 9%. It is noteworthy that this overshoot is indeed a manifestation of the Gibb’s phenomenon.

8.32 To transform a prototype low-pass filter into a bandstop filter of midband rejection frequency $\omega_0$ and bandwidth $B$, we may use the transformation

$$s \rightarrow \frac{Bs}{s^2 + \omega_0^2} \quad (1)$$

An as illustrative example, consider the low-pass filter:

$$H(s) = \frac{1}{s + 1} \quad (2)$$

Using the transformation of Eq. (1) in Eq. (2), we obtain a bandstop filter defined by

$$H(s) = \frac{1}{\frac{Bs}{s^2 + \omega_0^2} + 1}$$
An FIR filter of type 1 has an even length $M$ and is symmetric about $M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M - n] \quad \text{for} \quad n = 0, 1, \ldots, M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]e^{-jn\Omega}$$

which may be reformulated as follows:

$$H(e^{j\Omega}) = \sum_{n=0}^{M-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM/2} + \sum_{n=\frac{M}{2}+1}^{M} h[n]e^{-jn\Omega}$$

$$= \sum_{n=0}^{M-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM/2} + \sum_{n=0}^{M-1} h[M - n]e^{-j\Omega(M-n)}$$

$$= \sum_{n=0}^{M-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM/2} + \sum_{n=0}^{M-1} h[n]e^{-j\Omega(M-n)} \quad (1)$$

Define

$$a[k] = 2h\left[\frac{M}{2} - k\right], \quad k = 1, 2, \ldots, \frac{M}{2}$$

$$a[0] = h\left[\frac{M}{2}\right]$$

and let

$$n = \frac{M}{2} - k$$

We may then rewrite Eq. (1) in the equivalent form:
From Eq. (2) we may make the following observations for an FIR filter of type I:

1. The frequency response $H(e^{j\Omega})$ has a linear phase component exemplified by the exponential $e^{-jM\Omega/2}$.

2. At $\Omega = 0,$

$$H(e^{j0}) = \sum_{k=0}^{M/2} a[k]$$

At $\Omega = \pi,$

$$H(e^{j\pi}) = \sum_{k=0}^{M/2} a[k](-1)^{k+M/2}$$

The implications of these two results are that there are no restrictions on $H(e^{j\Omega})$ at $\Omega = 0$ and $\Omega = \pi$.

8.34 For an FIR filter of type II, the filter length $M$ is even and it is antisymmetric in that its coefficients satisfy the condition

$$h[n] = -h[M-n], \quad 0 \leq n \leq \frac{M}{2} - 1$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]e^{-jn\Omega}$$

which may be reformulated as follows:
Define

\[
H(e^{j\Omega}) = \sum_{n=0}^{M-1} h[n] e^{-j n \Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} + \sum_{n=M/2+1}^{M} h[n] e^{-j n \Omega}
\]

\[
= \sum_{n=0}^{M/2-1} h[n] e^{-j n \Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} + \sum_{n=0}^{M-1} h[M-n] e^{-j(M-n)\Omega} + \sum_{n=M/2+1}^{M} h[n] e^{-j(M-n)\Omega}
\]

(1)

and let

\[
a[k] = 2h\left[\frac{M}{2} - k\right], \quad k = 1, 2, \ldots, \frac{M}{2}
\]

\[
a[0] = h\left[\frac{M}{2}\right].
\]

We may then rewrite Eq. (1) in the equivalent form

\[
H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{n=0}^{M/2} h\left[\frac{M}{2} - k\right] e^{jk\Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} - e^{-jM\Omega/2} \sum_{k=1}^{M/2} h\left[\frac{M}{2} - k\right] e^{-jk\Omega}
\]

\[
= e^{-jM\Omega/2} \left\{ 2 \sum_{k=1}^{M/2} a[k] (e^{jk\Omega} - e^{-jk\Omega}) + a[0] \right\}
\]

\[
= e^{-jM\Omega/2} \left\{ j \sum_{k=1}^{M/2} a[k] \sin(k\Omega + a[0]) \right\}
\]

\[
= e^{-jM\Omega/2} \sum_{k=0}^{M/2} a[k] \sin(k\Omega)
\]

(2)

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of Type II

1. The phase response includes a linear component exemplified by the exponential \(e^{-jM\Omega/2}\).

2. At \(\Omega = 0\),
At $\Omega = \pi$, $\sin(k\pi) = 0$ for integer $k$ and therefore,

$$H(e^{j\pi}) = 0$$

8.35 An FIR filter of type III is characterized as follows:

- The filter length $M$ is an odd integer.
- The filter is symmetric about the noninteger midpoint $n = M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M - n] \quad \text{for } 0 \leq n \leq M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n] e^{-jn\Omega}$$

which may be reformulated as follows:

$$H(e^{j\Omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n] e^{-jn\Omega} + \sum_{n=\frac{M+1}{2}}^{M} h[n] e^{-jn\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n] e^{-jn\Omega} + \sum_{n=0}^{\frac{M-1}{2}} h[M-n] e^{-j(M-n)\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n] (e^{-jn\Omega} + e^{-j(M-n)\Omega})$$

$$= e^{-jM\Omega/2} \sum_{n=0}^{\frac{M-1}{2}} h[n] \left( e^{j\left(\frac{M}{2} - n\right)\Omega} + e^{-j\left(\frac{M}{2} - n\right)\Omega} \right)$$

Define

$$b[k] = 2h\left[\frac{M+1}{2} - k\right] \quad \text{for } k = 1, 2, \ldots, \frac{M+1}{2}$$

and let

$$e^{-jM\Omega/2}$$
We may then rewrite Eq. (1) in the equivalent form

\[ H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{k=1}^{M+1} \frac{1}{2} b(k) \left( e^{j\Omega\left(k - \frac{1}{2}\right)} + e^{-j\Omega\left(k - \frac{1}{2}\right)} \right) \]

\[ = e^{-jM\Omega/2} \sum_{k=1}^{M+1} b(k) \cos\left(\Omega\left(k - \frac{1}{2}\right)\right) \]

(2)

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of type III:

1. The phase response of the filter is linear as exemplified by the exponential factor \( e^{-jM\Omega/2} \).

2. At \( \Omega = 0 \),

\[ H(e^{j0}) = \sum_{k=1}^{M+1} b(k) \]

which shows that there is no restriction on \( H(e^{j0}) \).

At \( \Omega = \pi \),

\[ H(e^{j\pi}) = e^{-jM\pi/2} \sum_{k=1}^{M+1} b(k) \cos\left(\pi\left(k - \frac{1}{2}\right)\right) \]

\[ = e^{-jM\pi/2} \sum_{k=1}^{M+1} b(k) \sin(\pi k) \]

which is zero since \( \sin(\pi k) = 0 \) for all integer values of \( k \).

8.36 An FIR filter of type IV is characterized as follows:

- The filter length \( M \) is an odd integer.
- The filter is antisymmetric about the noninteger midpoint \( n = M/2 \) in that its coefficients satisfy the condition

\[ h[n] = -h[M - n] \quad \text{for } 0 \leq n \leq M \]

The frequency response of the filter is
which may be reformulated as follows:

\[
H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]e^{-jn\Omega}
\]

From Eq. (2) we may make the following observations on the FIR filter of type IV:

\[
b[k] = 2h\left[\frac{M+1}{2} - k\right] \quad \text{for } k = 1, 2, \ldots, \frac{M+1}{2}
\]

and let

\[
k = \frac{M+1}{2} - n
\]

We may then rewrite Eq. (1) in the equivalent form

\[
H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{k=1}^{M+1} h\left[\frac{M+1}{2} - k\right] \left( e^{j\left(k-\frac{1}{2}\right)\Omega} - e^{-j\left(k-\frac{1}{2}\right)\Omega} \right)
\]

\[
= je^{-jM\Omega/2} \sum_{k=1}^{M+1} 2h\left[\frac{M+1}{2} - k\right] \sin\left(\left(k-\frac{1}{2}\right)\Omega\right)
\]

\[
= je^{-jM\Omega/2} \sum_{k=1}^{M+1} b[k] \sin\left(\left(k-\frac{1}{2}\right)\Omega\right)
\]
1. The phase response of the filter includes a linear component exemplified by the exponential $e^{-jM\Omega/2}$.

2. At $\Omega = 0$,
   \[ H(e^{j0}) = 0 \]
   At $\Omega = \pi$,
   \[ H(e^{j\pi}) = j e^{-jM\Omega/2} \sum_{k=1}^{M+1} b(k) \sin\left(\left(k - \frac{1}{2}\right)\pi\right) \]
   \[ = j e^{-jM\Omega/2} \sum_{k=1}^{M+1} (-1)^{k+1} b[k] \]
   which shows that $H(e^{j\pi})$ can assume an arbitrary value.

8.37 The FIR digital filter used as a discrete-time differentiator in Example 8.6 exhibits the following properties:
- The filter length $M$ is an odd integer.
- The frequency response of the filter satisfies the conditions:
  1. At $\Omega = 0$,
     \[ H(e^{j0}) = 0 \]
  2. At $\Omega = \pi$,
     \[ H(e^{j\pi}) = 0 \]
   These properties are basic properties of an FIR filter of type III discussed in Problem 8.34. We therefore immediately deduce that the FIR filter of Example 8.6 is antisymmetric about the noninteger point $n = M/2$.

8.38 For a digital IIR filter, the transfer function $H(z)$ may be expressed as
   \[ H(z) = \frac{N(z)}{D(z)} \]
   where $N(z)$ and $D(z)$ are polynomials in $z^{-1}$. The filter is unstable if any pole of $H(z)$ or, equivalently, any zero of the denominator polynomial $D(z)$ lies outside the unit circle in the $z$-plane. According to the bilinear transform,
   \[ H(z) = \left. H_a(s) \right|_{s = \frac{z-1}{z+1}} \]
where $H_a(s)$ is the transfer function of an analog filter used as the basis for designing the digital IIR filter. The poles of $H(z)$ outside the unit circle in the $z$-plane correspond to certain poles of $H(s)$ in the right half of the $s$-plane. Conversely, the poles of $H(s)$ in the right half of the $s$-plane are mapped onto the outside of the unit circle in the $z$-plane. Now if any pole of $H_a(s)$ lies in the right half plane, the analog filter is unstable. Hence if any such filter is used in the bilinear transform, the resulting digital filter is likewise unstable.

8.39 We are given an analog filter whose transfer function is defined by

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-d_k}$$

Recall the Laplace transform pair

$$e^{dt} \Leftrightarrow \frac{1}{s-d_k}$$

It follows therefore that the impulse response of the analog filter is

$$h_a(t) = \sum_{k=1}^{N} A_k e^{d_k t}$$

(1)

According to the method of impulse invariance, the impulse response of a digital filter derived from the analog filter of Eq. (1) is defined by

$$h[n] = T_s h_a(nT_s)$$

where $T_s$ is the sampling period. Hence, from Eq. (1) we find that

$$h[n] = \sum_{k=1}^{N} T_s A_k e^{n d_k T_s}$$

(2)

Now recall the $z$-transform pair:

$$e^{n d_k T_s} \leftrightarrow \frac{1}{1-e^{d_k T_s} z^{-1}}$$

Hence, the transfer function of the digital filter is deduced from Eq. (2) to be

$$H(z) = \sum_{k=1}^{N} \frac{T_s A_k}{1-e^{d_k T_s} z^{-1}}$$

8.40 Consider a discrete-time system whose transfer function is denoted by $H(z)$. By definition,
\[ H(z) = \frac{Y(z)}{X(z)} \]

where \( Y(z) \) and \( X(z) \) are respectively the \( z \)-transforms of the output sequence \( y[n] \) and input sequence \( x[n] \). Let \( H_{eq}(z) \) denote the \( z \)-transform of the equalizer connected in cascade with \( H(z) \).

Let \( x'[n] \) denote the equalizer output in response to \( y[n] \) as the input. Ideally,
\[ x'[n] = x[n - n_0] \]

where \( n_0 \) is an integer delay. Hence,
\[
H_{eq}(z) = \frac{X'(z)}{Y(z)} = \frac{z^{-n_0}X(z)}{Y(z)} = \frac{z^{-n_0}}{H(z)}
\]

Putting \( z = e^{j\Omega} \), we may thus write
\[
H_{eq}(e^{j\Omega}) = \frac{e^{-jn_0\Omega}}{H(e^{j\Omega})}
\]

8.41 The phase delay of an FIR filter of even length \( M \) and antisymmetric impulse response is linear with frequency \( \Omega \) as shown by
\[
\Theta(\Omega) = -M\Omega
\]

Hence, such a filter used as an equalizer introduces a constant delay
\[
\tau(\Omega) = \frac{\partial \Theta(\Omega)}{\partial \Omega} = M \text{ samples}
\]

The implication of this result is that as we make the filter length \( M \) larger, the constant delay introduced by the equalizer is correspondingly increased. From a practical perspective, such a trend is highly undesirable.
Computer Experiments

% Solution to 8.42

b = fir1(22,1/3,hamming(23));

subplot(2,1,1)
plot(b);
title(’Impulse Response’)
ylabel(’Amplitude’)
xlabel(’Time (s)’)
grid

[H,w] = freqz(b,1,512,2*pi);
subplot(2,1,2)
plot(w,abs(H))
title(’Magnitude Response’)
ylabel(’Magnitude’)
xlabel(’Frequency (w)’)
grid

% Solution to problem 8.43

clear;

M = 100;
n = 0:M;
f = (n-M/2);
% Integration by parts (see example 8.6)
h = cos(pi*f)./f - sin(pi*f)./(pi*f.^2);
k = isnan(h);
h(k) = 0;
h_rect = h;
h_hamm = h .* hamming(length(h))';

[H,w] = freqz(h_rect,1,512,2*pi);

figure(1)
subplot(2,1,1)
plot(h_rect);
title('Rectangular Windowed Differentiator')
xlabel('Step')
ylabel('Amplitude')
grid
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
ylabel('Magnitude')
xlabel('Frequency (w)')
grid

[H,w] = freqz(h_hamm,1,512,2*pi);

figure(2);
subplot(2,1,1)
plot(h_hamm);
title('Hamming Windowed Differentiator')
xlabel('Step')
ylabel('Amplitude')
grid
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
ylabel('Magnitude')
xlabel('Frequency (w)')
grid
8.44 For a Butterworth low-pass filter of order $N$, the squared magnitude response is

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}$$

where $\omega_c$ is the cutoff frequency.

(a) We are given the following specifications:

(i) $\omega_c = 2\pi \times 800$ rad/s

(ii) At $\omega = 2\pi \times 1,500$ rad/s, we have
or, equivalently

\[ |H(j\omega)|^2 = \frac{1}{31.6228} \]

Substituting these values in Eq. (1):

\[ 31.6228 = 1 + \left( \frac{2\pi \times 1,200}{2\pi \times 800} \right)^{2N} \]

\[ = 1 + (1.5)^{2N} \]

Solving for the filter order \( N \):

\[ N = \frac{1}{2} \left( \log 30.6228 / \log 1.5 \right) \]

\[ = 4.2195 \]

So we choose \( N = 5 \).

---

% Solution to Problem 8.44

omegaC = 0.2;
N = 5;
wc = tan(omegaC/2);

coeff = [ 1 3.2361 5.2361 5.2361 3.2361 1];  
%(see table 8.1)

ns = wc^N;
ds = coeff .* (wc.^[0:N]);

[nz,dz]=bilinear(ns,ds,0.5);

[H,W]=freqz(nz,dz,512);
subplot(2,1,1)
plot(W,abs(H))
title('Magnitude Response of IIR low-pass filter')
xlabel('rad/s')
ylabel('Magnitude')
grid

phi = 180/pi *angle(H);
subplot(2,1,2)
plot(W,phi)
title('Phase Response')
xlabel('rad/s')
ylabel('degrees')
grid
% set(gcf,'name',['Low Pass: order=' num2str(N) ' wc=' num2str(omegaC)])

omegaC = 0.6;
N = 5;
wc = tan(omegaC/2);

coeff = [ 1 3.2361 5.2361 5.2361 3.2361 1];  %(see table 8.1)

ns = [1/wc^N zeros(1,N)];
ds = fliplr(coeff ./ (wc.^[0:N]));
[nz,dz]=bilinear(ns,ds,0.5);

[H,W]=freqz(nz,dz,512);
subplot(2,1,1)
plot(W,abs(H))
title('Magnitude Response of IIR high-pass filter')
xlabel('rad/s')
ylabel('Magnitude')
grid

phi = 180/pi *angle(H);
subplot(2,1,2)
plot(W,phi)
title('Phase Response')
xlabel('rad/s')
ylabel('degrees')
Solution to problem 8.46

\[
w = 0:pi/511:pi;
\]
\[
den = sqrt(1+2*((w/pi).^2) + (w/pi).^4);
\]
\[
Hchan = 1./den;
\]
\[
taps = 95;
\]
\[
M = taps -1;
\]
\[
n = 0:M;
\]
\[
f = n-M/2;
\]

%Term 1
\[
hh1 = fftshift(ifft(ones(taps,1)).*hamming(taps));
\]

%Term 2
\[
h = cos(pi*f)./f - sin(pi*f)./(pi*f.^2);
\]
\[
k=isnan(h); h(k)=0;
\]
\[
hh2 = 2*(hamming(taps).^'.*h)/pi;
\]

%Term 3
\[
hh3a = -(1./(pi*f)).* sin(pi*f);
\]
\[
hh3b = -(2./(pi*f).^2).* cos(pi*f);
\]
\[
hh3c = (2./(pi*f).^3).* sin(pi*f);
\]
\[
hh3 = hamming(taps).^'.* (hh3a + hh3b + hh3c);
\]
\[
hh = hh1' + hh2 + hh3';
\]
\[
hh(48)=2/3;
\]
\[ \text{[Heq,w]} = \text{freqz(hh,1,512,2*pi)}; \]

\[ p = 0.7501; \]

\[ \text{Hcheq} = (p*\text{abs(Heq)}).*\text{Hchan}; \]

\[ \text{plot}(w,p*\text{abs(Heq)},'b--') \]

\[ \text{hold on} \]

\[ \text{plot}(w,\text{abs(Hchan)},'g-.') \]

\[ \text{plot}(w,\text{abs(Hcheq)},'r-') \]

\[ \text{legend('Heq', 'Hcan', 'Hcheq',1)} \]

\[ \text{hold off} \]

\[ \text{xlabel('Frequency (\Omega)')} \]

\[ \text{ylabel('Magnitude Response')} \]