Solutions to Skill-Assessment Exercises

To Accompany

Control Systems Engineering
4th Edition

By

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John Wiley & Sons

ISBN 0-471-44577-0

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

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Chapter 2

2.1.

The Laplace transform of t is $\frac{1}{s^2}$ using Table 2.1, Item 3. Using Table 2.2, Item 4,

$$F(s) = \frac{1}{(s+5)^2}.$$

2.2.

Expanding F(s) by partial fractions yields:

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)^2} + \frac{D}{(s+3)}$$

where,

$$A = \left.\frac{10}{(s+2)(s+3)^2}\right|_{S\to 0} = \frac{5}{9} \quad B = \left.\frac{10}{s(s+3)^2}\right|_{S\to -2} = -5 \quad C = \left.\frac{10}{s(s+2)}\right|_{S\to -3} = \frac{10}{3}, \text{ and}$$

$$D = (s+3)^2 \left.\frac{dF(s)}{ds}\right|_{s\to -3} = \frac{40}{9}$$

Taking the inverse Laplace transform yields,

$$f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$$

2.3.

Taking the Laplace transform of the differential equation assuming zero initial conditions yields:

$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$

Collecting terms,

$$(s^3 + 3s^2 + 7s + 5)C(s) = (s^2 + 4s + 3)R(s)$$

Thus,

$$\frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

2.4.

$$G(s) = \frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + 6s + 2}$$

Cross multiplying yields,

$$\frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 2c = 2\frac{dr}{dt} + r$$

2.5.

$$C(s) = R(s)G(s) = \frac{1}{s^2} * \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

where

$$A = \left.\frac{1}{(s+4)(s+8)}\right|_{S\to 0} = \frac{1}{32}\ B = \left.\frac{1}{s(s+8)}\right|_{S\to -4} = -\frac{1}{16},\ \text{and}\ C = \left.\frac{1}{s(s+4)}\right|_{S\to -8} = \frac{1}{32}$$

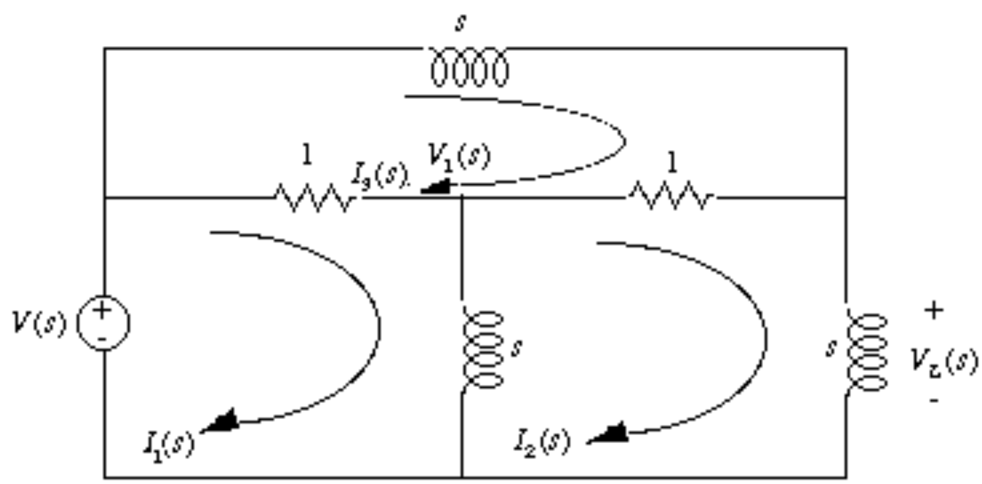
Thus,

$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

2.6.

Mesh Analysis

Transforming the network yields,



Now, writing the mesh equations,

$$(s+1)I_1(s) - sI_2(s) - I_3(s) = V(s)$$
$$-sI_1(s) + (2s+1)I_2(s) - I_3(s) = 0$$
$$-I_1(s) - I_2(s) + (s+2)I_3(s) = 0$$

Solving the mesh equations for $I_2(s)$,

$$I_2(s) = \frac{\begin{vmatrix} (s+1) & V(s) & -1 \\ -s & 0 & -1 \\ -1 & 0 & (s+2) \end{vmatrix}}{\begin{vmatrix} (s+1) & -s & -1 \\ -s & (2s+1) & -1 \\ -1 & -1 & (s+2) \end{vmatrix}} = \frac{(s^2+2s+1)V(s)}{s(s^2+5s+2)}$$

But, $V_L(s) = sI_2(s)$

Hence,

$$V_L(s) = \frac{(s^2+2s+1)V(s)}{(s^2+5s+2)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2+2s+1}{s^2+5s+2}$$

Nodal Analysis

Writing the nodal equations,

$$(\frac{1}{s}+2)V_1(s) - V_L(s) = V(s)$$
$$-V_1(s) + (\frac{2}{s}+1)V_L(s) = \frac{1}{s}V(s)$$

Solving for $V_L(s)$,

$$V_L(s) = \frac{\begin{vmatrix} (\frac{1}{s}+2) & V(s) \\ -1 & \frac{1}{s}V(s) \end{vmatrix}}{\begin{vmatrix} (\frac{1}{s}+2) & -1 \\ -1 & (\frac{2}{s}+1) \end{vmatrix}} = \frac{(s^2+2s+1)V(s)}{(s^2+5s+2)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2+2s+1}{s^2+5s+2}$$

2.7.

Inverting

$$G(s)=-\frac{Z_2(s)}{Z_1(s)}=\frac{-100000}{(10^5/s)}=-s$$

Noninverting

$$G(s)=\frac{[Z_1(s)+Z_2(s)]}{Z_1(s)}=\frac{(\frac{10^5}{s}+10^5)}{(\frac{10^5}{s})}=s+1$$

2.8.

Writing the equations of motion,

$$(s^2+3s+1)X_1(s)-(3s+1)X_2(s)=F(s)$$

$$-(3s+1)X_1(s)+(s^2+4s+1)X_2(s)=0$$

Solving for $X_2(s)$,

$$X_2(s)=\frac{\begin{vmatrix}(s^2+3s+1) & F(s)\\ -(3s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+3s+1) & -(3s+1)\\ -(3s+1) & (s^2+4s+1)\end{vmatrix}}=\frac{(3s+1)F(s)}{s(s^3+7s^2+5s+1)}$$

Hence,

$$\frac{X_2(s)}{F(s)}=\frac{(3s+1)}{s(s^3+7s^2+5s+1)}$$

2.9.

Writing the equations of motion,

$$(s^2+s+1)\theta_1(s)-(s+1)\theta_2(s)=T(s)$$

$$-(s+1)\theta_1(s)+(2s+2)\theta_2(s)=0$$

where $\theta_1(s)$ is the angular displacement of the inertia.

Solving for $\theta_2(s)$,

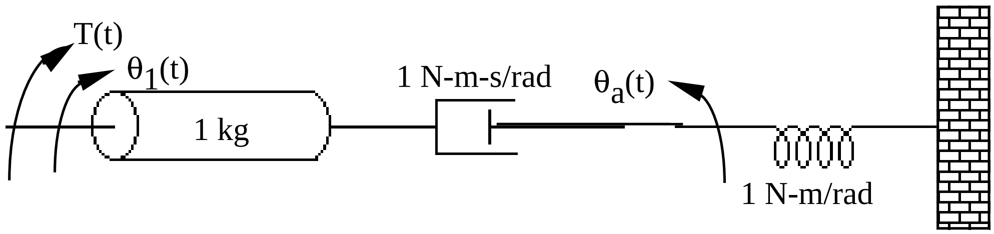
$$\theta_2(s)=\frac{\begin{vmatrix}(s^2+s+1) & T(s)\\ -(s+1) & 0\end{vmatrix}}{\begin{vmatrix}(s^2+s+1) & -(s+1)\\ -(s+1) & (2s+2)\end{vmatrix}}=\frac{(s+1)F(s)}{2s^3+3s^2+2s+1}$$

From which, after simplification,

$$\theta_2(s) = \frac{1}{2s^2 + s + 1}$$

2.10.

Transforming the network to one without gears by reflecting the 4 N-m/rad spring to the left and multiplying by $(25/50)^2$, we obtain,



Writing the equations of motion,

$$(s^2 + s)\theta_1(s) - s\theta_a(s) = T(s)$$
$$-s\theta_1(s) + (s+1)\theta_a(s) = 0$$

where $\theta_1(s)$ is the angular displacement of the 1-kg inertia.

Solving for $\theta_a(s)$,

$$\theta_a(s) = \frac{\begin{vmatrix} (s^2+s) & T(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s) & -s \\ -s & (s+1) \end{vmatrix}} = \frac{sT(s)}{s^3 + s^2 + s}$$

From which,

$$\frac{\theta_a(s)}{T(s)} = \frac{1}{s^2 + s + 1}$$

But, $\theta_2(s) = \frac{1}{2}\theta_a(s)$.

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{1/2}{s^2 + s + 1}$$

2.11.

First find the mechanical constants.

$$J_m = J_a + J_L\left(\frac{1}{5} * \frac{1}{4}\right)^2 = 1 + 400\left(\frac{1}{400}\right) = 2$$

$$D_m = D_a + D_L\left(\frac{1}{5} * \frac{1}{4}\right)^2 = 5 + 800\left(\frac{1}{400}\right) = 7$$

Now find the electrical constants. From the torque-speed equation, set $\omega_m = 0$ to find stall torque and set $T_m = 0$ to find no-load speed. Hence,

$$T_{stall} = 200$$

$$\omega_{no-load} = 25$$

which,

$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{200}{100} = 2$$

$$K_b = \frac{E_a}{\omega_{no-load}} = \frac{100}{25} = 4$$

Substituting all values into the motor transfer function,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_T}{R_a J_m}}{s(s + \frac{1}{J_m}(D_m + \frac{K_T K_b}{R_a})} = \frac{1}{s(s + \frac{15}{2})}$$

where $\theta_m(s)$ is the angular displacement of the armature.

Now $\theta_L(s) = \frac{1}{20}\theta_m(s)$. Thus,

$$\frac{\theta_L(s)}{E_a(s)} = \frac{1/20}{s(s + \frac{15}{2})})$$

2.12.

Letting

$$\theta_1(s) = \omega_1(s)/s$$

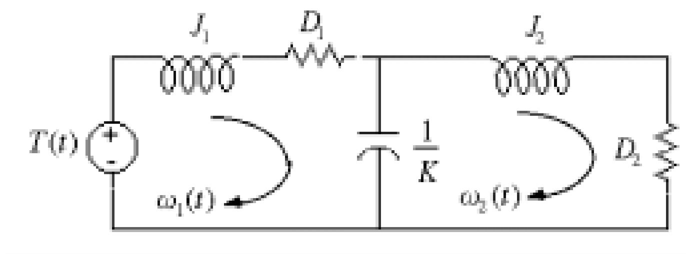
$$\theta_2(s) = \omega_2(s)/s$$

in Eqs. 2.127, we obtain

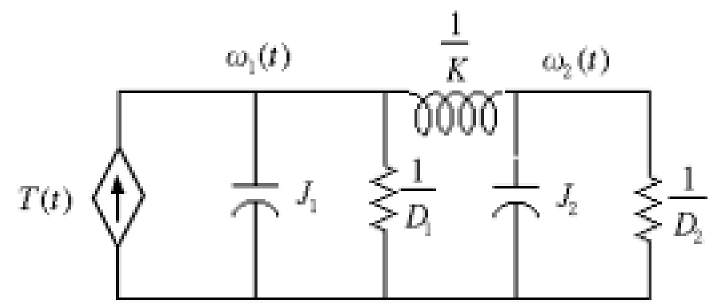
$$(J_1 s + D_1 + \frac{K}{s})\omega_1(s) - \frac{K}{s}\omega_2(s) = T(s)$$

$$-\frac{K}{s}\omega_1(s) + (J_2 s + D_2 + \frac{K}{s})\omega_2(s)$$

From these equations we can draw both series and parallel analogs by considering these to be mesh or nodal equations, respectively.



Series analog



Parallel analog

2.13.

Writing the nodal equation,

$$C\frac{dv}{dt} + i_r - 2 = i(t)$$

But,

$$C = 1$$

$$v = v_o + \delta v$$

$$i_r = e^{v_r} = e^v = e^{v_o + \delta v}$$

Substituting these relationships into the differential equation,

$$\frac{d(v_o + \delta v)}{dt} + e^{v_o + \delta v} - 2 = i(t) \qquad (1)$$

We now linearize e^v.

The general form is

$$f(v) - f(v_o) \approx \left.\frac{df}{dv}\right|_{v_o} \delta v$$

Substituting the function, $f(v) = e^v$, with $v = v_o + \delta v$ yields,

$$e^{v_o + \delta v} - e^{v_o} \approx \left.\frac{de^v}{dv}\right|_{v_o} \delta v$$

Solving for $e^{v_o + \delta v}$,

$$e^{v_o+\delta v} = e^{v_o} + \left.\frac{de^v}{dv}\right|_{v_o} \delta v = e^{v_o} + e^{v_o}\delta v$$

Substituting into Eq. (1)

$$\frac{d\delta v}{dt} + e^{v_o} + e^{v_o}\delta v - 2 = i(t) \quad (2)$$

Setting $i(t) = 0$ and letting the circuit reach steady state, the capacitor acts like an open circuit. Thus, $v_o = v_r$ with $i_r = 2$. But, $i_r = e^{v_r}$ or $v_r = \ln i_r$.

Hence, $v_o = \ln 2 = 0.693$. Substituting this value of v_o into Eq. (2) yields

$$\frac{d\delta v}{dt} + 2\delta v = i(t)$$

Taking the Laplace transform,

$$(s+2)\delta v(s) = I(s)$$

Solving for the transfer function, we obtain

$$\frac{\delta v(s)}{I(s)} = \frac{1}{s+2}$$

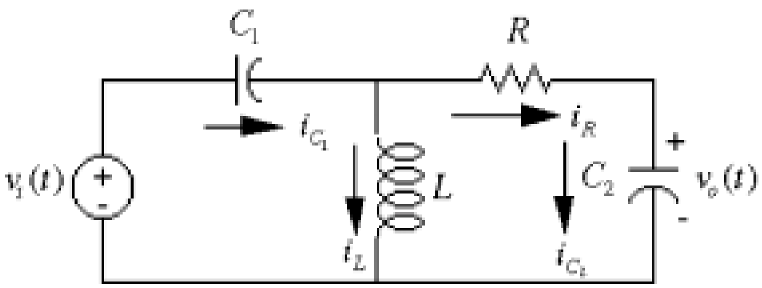
or

$$\frac{V(s)}{I(s)} = \frac{1}{s+2}$$ about equilibrium.

Chapter 3

3.1.

Identifying appropriate variables on the circuit yields



Writing the derivative relations

$$
\begin{aligned}
&C_1 \frac{dv_{C_1}}{dt} = i_{C_1} \\
&L \frac{di_L}{dt} = v_L \qquad\qquad (1) \\
&C_2 \frac{dv_{C_2}}{dt} = i_{C_2}
\end{aligned}
$$

Using Kirchhoff's current and voltage laws,

$$
\begin{aligned}
&i_{C_1} = i_L + i_R = i_L + \frac{1}{R}(v_L - v_{C_2}) \\
&v_L = -v_{C_1} + v_i \\
&i_{C_2} = i_R = \frac{1}{R}(v_L - v_{C_2})
\end{aligned}
$$

Substituting these relationships into Eqs. (1) and simplifying yields the state equations as

$$
\begin{aligned}
&\frac{dv_{C_1}}{dt} = -\frac{1}{RC_1} v_{C_1} + \frac{1}{C_1} i_L - \frac{1}{RC_1} v_{C_2} + \frac{1}{RC_1} v_i \\
&\frac{di_L}{dt} = -\frac{1}{L} v_{C_1} + \frac{1}{L} v_i \\
&\frac{dv_{C_2}}{dt} = -\frac{1}{RC_2} v_{C_1} - \frac{1}{RC_2} v_{C_2} \frac{1}{RC_2} v_i
\end{aligned}
$$

where the output equation is

$$v_o = v_{C_2}$$

Putting the equations in vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & -\frac{1}{RC_1} \\ -\frac{1}{L} & 0 & 0 \\ -\frac{1}{RC_2} & 0 & -\frac{1}{RC_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{L} \\ \frac{1}{RC_2} \end{bmatrix} v_i(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

3.2.

Writing the equations of motion

$$\begin{aligned} (s^2+s+1)X_1(s) \quad & -sX_2(s) & & = F(s) \\ -sX_1(s) + (s^2+s+1)X_2(s) & & -X_3(s) & = 0 \\ & -X_2(s) + (s^2+s+1)X_3(s) & & = 0 \end{aligned}$$

Taking the inverse Laplace transform and simplifying,

$$\ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f$$

$$\ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3$$

$$\ddot{x}_3 = -\dot{x}_3 - x_3 + x_2$$

Defining state variables, z_i,

$$z_1 = x_1;\ z_2 = \dot{x}_1;\ z_3 = x_2;\ z_4 = \dot{x}_2;\ z_5 = x_3;\ z_6 = \dot{x}_3$$

Writing the state equations using the definition of the state variables and the inverse transform of the differential equation,

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f = -z_2 - z_1 + z_4 + f$$

$$\dot{z}_3 = \dot{x}_2 = z_4$$

$$\dot{z}_4 = \ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3 = z_2 - z_4 - z_3 + z_5$$

$$\dot{z}_5 = \dot{x}_3 = z_6$$

$$\dot{z}_6 = \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2 = -z_6 - z_5 + z_3$$

The output is z_5. Hence, $y = z_5$. In vector-matrix form,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t);\ \mathrm{y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{z}$$

3.3.

First derive the state equations for the transfer function without zeros.

$$\frac{X(s)}{R(s)} = \frac{1}{s^2 + 7s + 9}$$

Cross multiplying yields

$$(s^2 + 7s + 9)X(s) = R(s)$$

Taking the inverse Laplace transform assuming zero initial conditions, we get

$$\ddot{x} + 7\dot{x} + 9x = r$$

Defining the state variables as,

$$x_1 = x$$

$$x_2 = \dot{x}$$

Hence,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = -7\dot{x} - 9x + r = -9x_1 - 7x_2 + r$$

Using the zeros of the transfer function, we find the output equation to be,

$$c = 2\dot{x} + x = x_1 + 2x_2$$

Putting all equation in vector-matrix form yields,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$c = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}$$

3.4.

The state equation is converted to a transfer function using

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \qquad (1)$$

where

$\mathbf{A}=\begin{bmatrix}-4 & -1.5\\ 4 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}2\\ 0\end{bmatrix}$, and $\mathbf{C}=\begin{bmatrix}1.5 & 0.625\end{bmatrix}$.

Evaluating $(s\mathbf{I}-\mathbf{A})$ yields

$$(s\mathbf{I}-\mathbf{A})=\begin{bmatrix}s+4 & 1.5\\ -4 & s\end{bmatrix}$$

Taking the inverse we obtain

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^2+4s+6}\begin{bmatrix}s & -1.5\\ 4 & s+4\end{bmatrix}$$

Substituting all expressions into Eq. (1) yields

$$G(s)=\frac{3s+5}{s^2+4s+6}$$

3.5.

Writing the differential equation we obtain

$$\frac{d^2x}{dt^2}+2x^2=10+\delta f(t) \qquad (1)$$

Letting $x=x_o+\delta x$ and substituting into Eq. (1) yields

$$\frac{d^2(x_o+\delta x)}{dt^2}+2(x_o+\delta x)^2=10+\delta f(t) \qquad (2)$$

Now, linearize x^2.

$$(x_o+\delta x)^2-x_o^{\,2}=\left.\frac{d(x^2)}{dx}\right|_{x_o}\delta x=2x_o\delta x$$

from which

$$(x_o+\delta x)^2=x_o^{\,2}+2x_o\delta x \qquad (3)$$

Substituting Eq. (3) into Eq. (1) and performing the indicated differentiation gives us the linearized intermediate differential equation,

$$\frac{d^2\delta x}{dt^2}+4x_o\delta x=-2x_o^{\,2}+10+\delta f(t) \qquad (4)$$

The force of the spring at equilibrium is 10 N. Thus, since $F=2x^2$,

$$10=2x_o^{\,2}$$

from which

$$x_o=\sqrt{5}$$

Substituting this value of x_o into Eq. (4) gives us the final linearized differential equation.

$$\frac{d^2\delta x}{dt^2} + 4\sqrt{5}\delta x = \delta f(t)$$

Selecting the state variables,

$$x_1 = \delta x$$

$$x_2 = \dot{\delta x}$$

Writing the state and output equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{\delta x} = -4\sqrt{5}x_1 + \delta f(t)$$

$$y = x_1$$

Converting to vector-matrix form yields the final result as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4\sqrt{5} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta f(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

Chapter 4

4.1.

For a step input

$$C(s) \rfloor \; \frac{10(s\ )\ 4)(s\ )\ 6)}{s(s\ )\ 1)(s\ )\ 7)(s\ )\ 8)(s\ )\ 10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+7} + \frac{D}{s+8} + \frac{E}{s+10}$$

Taking the inverse Laplace transform,

$$c(t) = A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$$

4.2.

Since $a = 50$, $T_c = \frac{1}{a} = \frac{1}{50} = 0.02\,\text{s}$; $T_s = \frac{4}{a} = \frac{4}{50} = 0.08$ s; and $T_r = \frac{2.2}{a} = \frac{2.2}{50} = 0.044$ s.

4.3.

a. Since poles are at $-6 \pm j19.08$, $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$.

b. Since poles are at –78.54 and –11.46, $c(t) = A + Be^{-78.54t} + Ce^{-11.4t}$.

c. Since poles are double on the real axis at –15 $c(t) = A + Be^{-15t} + Cte^{-15t}$.

d. Since poles are at $\pm j25$, $c(t) = A + B\cos(25t + \phi)$.

4.4.

a. $\omega_n = \sqrt{400} = 20$ and $2\zeta\omega_n = 12$; $\therefore \zeta = 0.3$ and system is underdamped.

b. $\omega_n = \sqrt{900} = 30$ and $2\zeta\omega_n = 90$; $\therefore \zeta = 1.5$ and system is overdamped.

c. $\omega_n = \sqrt{225} = 15$ and $2\zeta\omega_n = 30$; $\therefore \zeta = 1$ and system is critically damped.

d. $\omega_n = \sqrt{625} = 25$ and $2\zeta\omega_n = 0$; $\therefore \zeta = 0$ and system is undamped.

4.5.

$\omega_n = \sqrt{361} = 19$ and $2\zeta\omega_n = 16$; $\therefore \zeta = 0.421$.

Now, $\text{T}_\text{s} = \frac{4}{\zeta\omega_n} = 0.5\ s$ and $\text{T}_\text{p} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.182$ s.

From Figure 4.16, $\omega_n \text{T}_\text{r} = 1.4998$. Therefore, $T_r = 0.079$ s.

Finally, $\%\text{os} = \text{e}^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 = 23.3\%$

4.6.

a. The second-order approximation is valid, since the dominant poles have a real part of –2 and the higher-order pole is at –15, i.e. more than five-times further.

b. The second-order approximation is not valid, since the dominant poles have a real part of –1 and the higher-order pole is at –4, i.e. not more than five-times further.

4.7.

a. Expanding $G(s)$ by partial fractions yields $G(s)=\frac{1}{s}+\frac{0.8942}{s+20}-\frac{1.5918}{s+10}-\frac{0.3023}{s+6.5}$.

But –0.3023 is not an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is not valid.

b. Expanding $G(s)$ by partial fractions yields $G(s)=\frac{1}{s}+\frac{0.9782}{s+20}-\frac{1.9078}{s+10}-\frac{0.0704}{s+6.5}$.

But 0.0704 is an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is valid.

4.8.

See Figure 4.31 in the textbook for the Simulink block diagram and the output responses.

4.9.

a. Since $s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}$, $(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}$. Also,

$\mathbf{BU}(s)=\begin{bmatrix} 0 \\ 1/(s+1) \end{bmatrix}$.

The state vector is $\mathbf{X}(s)=(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0)+\mathbf{BU}(s)]=\frac{1}{(s+1)(s+2)(s+3)}\begin{bmatrix} 2(s^2+7s+7) \\ s^2-4s-6 \end{bmatrix}$.

The output is $Y(s)=[1 \quad 3]\mathbf{X}(s)=\frac{5s^2+2s-4}{(s+1)(s+2)(s+3)}=-\frac{0.5}{s+1}-\frac{12}{s+2}+\frac{17.5}{s+3}$.

Taking the inverse Laplace transform yields $y(t)=-0.5e^{-t}-12e^{-2t}+17.5e^{-3t}$.

b. The eigenvalues are given by the roots of $|s\mathbf{I}-\mathbf{A}|=s^2+5s+6$, or –2 and –3.

4.10.

a. Since $(s\mathbf{I}-\mathbf{A})=\begin{bmatrix} s & -2 \\ 2 & s+5 \end{bmatrix}$, $(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^2+5s+4}\begin{bmatrix} s+5 & 2 \\ -2 & s \end{bmatrix}$. Taking the Laplace transform of each term, the state transition matrix is given by

$$\Phi(t)=\begin{bmatrix} \frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t}-\frac{2}{3}e^{-4t} \\ -\frac{2}{3}e^{-t}+\frac{2}{3}e^{-4t} & -\frac{1}{3}e^{-t}+\frac{4}{3}e^{-4t} \end{bmatrix}.$$

b. Since $\Phi(t-\tau)=\begin{bmatrix} \frac{4}{3}e^{-(t-\tau)}-\frac{1}{3}e^{-4(t-\tau)} & \frac{2}{3}e^{-(t-\tau)}-\frac{2}{3}e^{-4(t-\tau)} \\ -\frac{2}{3}e^{-(t-\tau)}+\frac{2}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)}+\frac{4}{3}e^{-4(t-\tau)} \end{bmatrix}$ and $\mathbf{Bu}(\tau)=\begin{bmatrix} 0 \\ e^{-2\tau} \end{bmatrix}$,

$$\Phi(t-\tau)\mathbf{Bu}(\tau)=\begin{bmatrix} \frac{2}{3}e^{-\tau}e^{-t}-\frac{2}{3}e^{2\tau}e^{-4t} \\ -\frac{1}{3}e^{-\tau}e^{-t}+\frac{4}{3}e^{2\tau}e^{-4t} \end{bmatrix}.$$

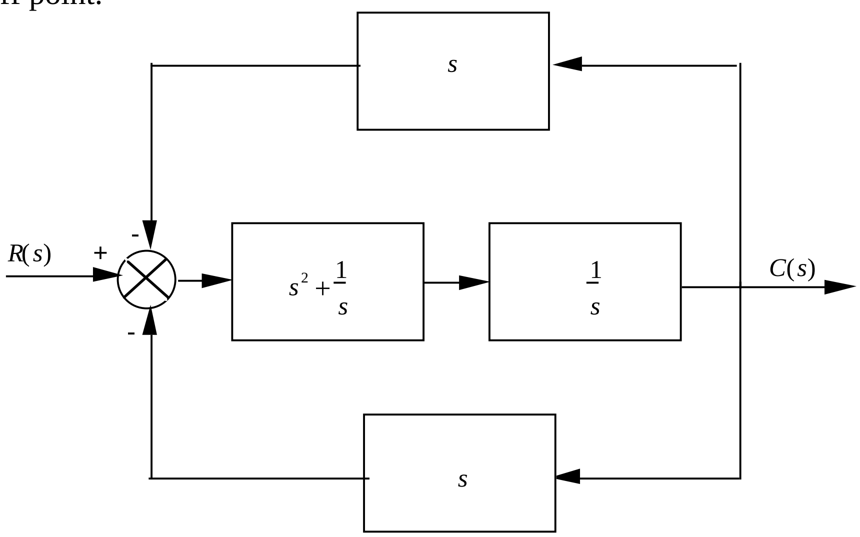
Thus, $\mathbf{x}(t)=\Phi(t)\mathbf{x}(0)+\int_0^t \Phi(t-\tau)\mathbf{Bu}(\tau)d\tau=\begin{bmatrix} \frac{10}{3}e^{-t}-e^{-2t}-\frac{4}{3}e^{-4t} \\ -\frac{5}{3}e^{-t}+e^{-2t}+\frac{8}{3}e^{-4t} \end{bmatrix}$.

c. $y(t)=\begin{bmatrix} 2 & 1 \end{bmatrix}\mathbf{x}=5e^{-t}-e^{-2t}$

Chapter 5

5.1.

Combine the parallel blocks in the forward path. Then, push $\frac{1}{s}$ to the left past the pickoff point.



Combine the parallel feedback paths and get 2s. Then, apply the feedback formula, simplify, and get, $T(s)=\dfrac{s^3+1}{2s^4+s^2+2s}$.

5.2.

Find the closed-loop transfer function, $T(s)=\dfrac{G(s)}{1+G(s)H(s)}=\dfrac{16}{s^2+as+16}$,

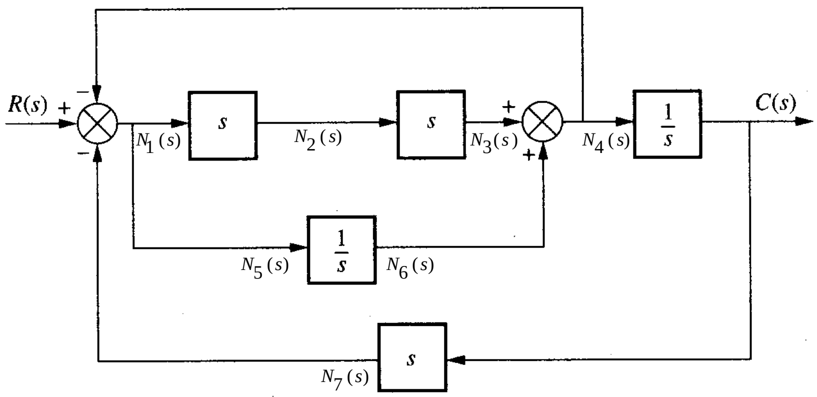
where $G(s)=\dfrac{16}{s(s+a)}$ and H(s) = 1. Thus, $\omega_n=4$ and $2\zeta\omega_n=a$, from which

$\zeta=\dfrac{a}{8}$. But, for 5% overshoot, $\zeta=\dfrac{-\ln(\frac{\%}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%}{100})}}$ = 0.69. Since, $\zeta=\dfrac{a}{8}$,

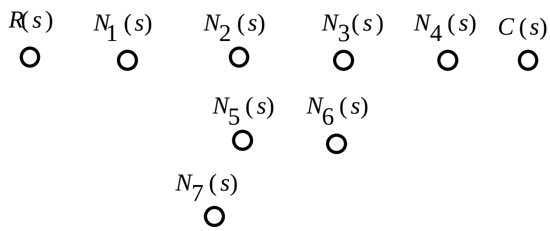
$a=5.52$.

5.3.

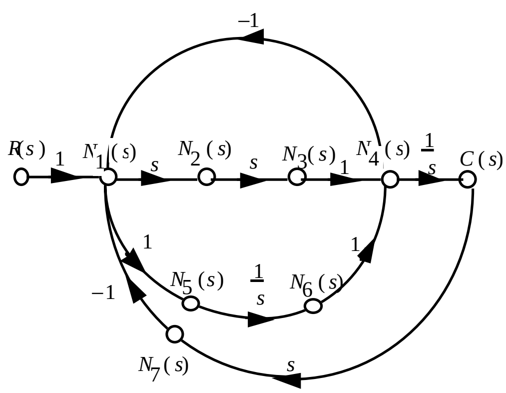
Label nodes.



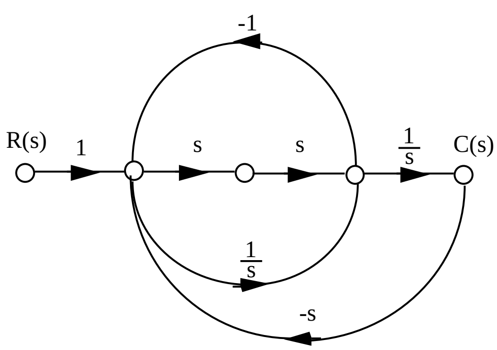
Draw nodes.



Connect nodes and label subsystems.



Eliminate unnecessary nodes.



5.4.

Forward-path gains are $G_1G_2G_3$ and G_1G_3.

Loop gains are $-G_1G_2H_1$, $-G_2H_2$, and $-G_3H_3$.

Nontouching loops are $[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$

and $[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3$.

Also, $\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3$.

Finally, $\Delta_1 = 1$ and $\Delta_2 = 1$.

Substituting these values into $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{\sum_k T_k\Delta_k}{\Delta}$ yields

$$T(s) = \frac{G_1(s)G_3(s)[1+G_2(s)]}{[1+G_2(s)H_2(s)+G_1(s)G_2(s)H_1(s)][1+G_3(s)H_3(s)]}$$

5.5.

The state equations are,

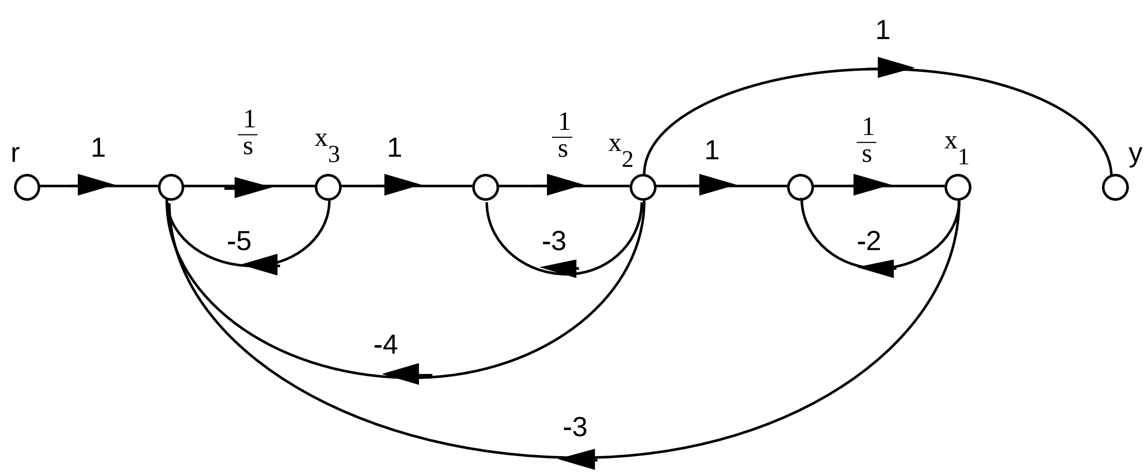
$$\dot{x}_1 = -2x_1 + x_2$$

$$\dot{x}_2 = -3x_2 + x_3$$

$$\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r$$

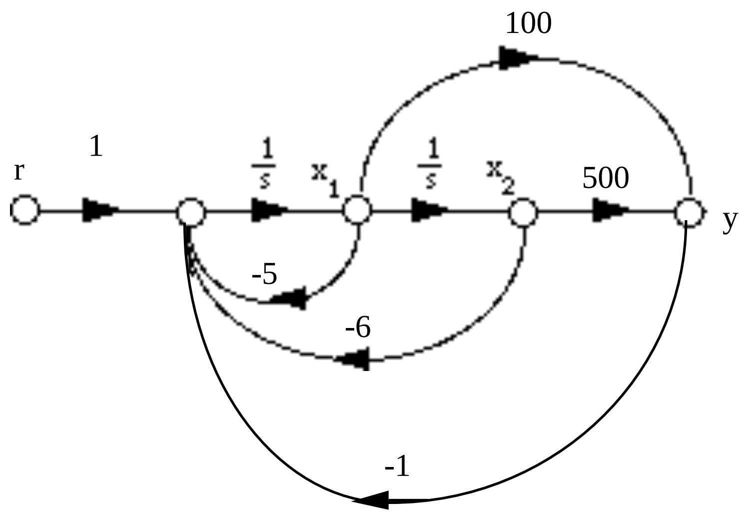
$$y = x_2$$

Drawing the signal-flow diagram from the state equations yields



5.6.

From $G(s) = \dfrac{100(s+5)}{s^2+5s+6}$ we draw the signal-flow graph in controller canonical form and add the feedback.



Writing the state equations from the signal-flow diagram, we obtain

$$\dot{\mathbf{x}} = \begin{bmatrix} -105 & -506 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 100 & 500 \end{bmatrix}\mathbf{x}$$

5.7.

From the transformation equations,

$$\mathbf{P}^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}$$

Taking the inverse,

$$\mathbf{P} = \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}$$

Now,

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$$

$$\mathbf{CP} = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = \begin{bmatrix} 0.8 & -1.4 \end{bmatrix}$$

Therefore,

$$\dot{\mathbf{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.8 & -1.4 \end{bmatrix}\mathbf{z}$$

5.8.

First find the eigenvalues.

$$|\lambda\mathbf{I}-\mathbf{A}| = \left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & 3\\ -4 & -6\end{bmatrix}\right| = \begin{vmatrix}\lambda-1 & -3\\ 4 & \lambda+6\end{vmatrix} = \lambda^2+5\lambda+6$$

From which the eigenvalues are –2 and –3.

Now use $\mathbf{Ax_i} = \lambda\mathbf{x_i}$ for each eigenvalue, λ. Thus,

$$\begin{bmatrix}1 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \lambda\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

For $\lambda = -2$,

$$3x_1 + 3x_2 = 0$$
$$-4x_1 - 4x_2 = 0$$

Thus $x_1 = -x_2$

For $\lambda = -3$

$$4x_1 + 3x_2 = 0$$
$$-4x_1 - 3x_2 = 0$$

Thus $x_1 = -x_2$ and $x_1 = -0.75x_2$; from which we let

$$\mathbf{P} = \begin{bmatrix}0.707 & -0.6\\ -0.707 & 0.8\end{bmatrix}$$

Taking the inverse yields

$$\mathbf{P}^{-1} = \begin{bmatrix}5.6577 & 4.2433\\ 5 & 5\end{bmatrix}$$

Hence,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix}5.6577 & 4.2433\\ 5 & 5\end{bmatrix}\begin{bmatrix}1 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}0.707 & -0.6\\ -0.707 & 0.8\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix}5.6577 & 4.2433\\ 5 & 5\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix} = \begin{bmatrix}18.39\\ 20\end{bmatrix}$$

$$\mathbf{CP} = \begin{bmatrix}1 & 4\end{bmatrix}\begin{bmatrix}0.707 & -0.6\\ -0.707 & 0.8\end{bmatrix} = \begin{bmatrix}-2.121 & 2.6\end{bmatrix}$$

Finally,

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}u$$

$$y = \begin{bmatrix} -2.121 & 2.6 \end{bmatrix}\mathbf{z}$$

Chapter 6

6.1.

Make a Routh table.

s^7	3	6	7	2
s^6	9	4	8	6
s^5	4.666666667	4.333333333	0	0
s^4	-4.35714286	8	6	0
s^3	12.90163934	6.426229508	0	0
s^2	10.17026684	6	0	0
s^1	-1.18515742	0	0	0
s^0	6	0	0	0

Since there are four sign changes and no complete row of zeros, there are four right half-plane poles and three left half-plane poles.

6.2.

Make a Routh table. We encounter a row of zeros on the s^3 row. The even polynomial is contained in the previous row as $-6s^4 + 0s^2 + 6$. Taking the derivative yields $-24s^3 + 0s$. Replacing the row of zeros with the coefficients of the derivative yields the s^3 row. We also encounter a zero in the first column at the s^2 row. We replace the zero with ε and continue the table. The final result is shown now as

s^6	1	-6	-1	6	
s^5	1	0	-1	0	
s^4	-6	0	6	0	
s^3	-24	0	0	0	ROZ
s^2	ε	6	0	0	
s^1	144/ε	0	0	0	
s^0	6	0	0	0	

There is one sign change below the even polynomial. Thus the even polynomial (4^{th} order) has one right half-plane pole, one left half-plane pole, and 2 imaginary axis poles. From the top of the table down to the even polynomial yields one sign change. Thus, the rest of the polynomial has one right half-plane root, and one left half-plane root. The total for the system is two right half-plane poles, two left half-plane poles, and 2 imaginary poles.

6.3.

Since $G(s) = \dfrac{K(s+20)}{s(s+2)(s+3)}$, $T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{K(s+20)}{s^3+5s^2+(6+K)s+20K}$

Form the Routh table.

s^3	1	$(6+K)$
s^2	5	$20K$
s^1	$\dfrac{30-15K}{5}$	
s^0	$20K$	

From the s^1 row, $K < 2$. From the s^0 row, $K > 0$. Thus, for stability, $0 < K < 2$.

6.4.

First find

$$|s\mathbf{I}-\mathbf{A}| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix}\right| = \begin{vmatrix} (s-2) & -1 & -1 \\ -1 & (s-7) & -1 \\ 3 & -4 & (s+5) \end{vmatrix} = s^3 - 4s^2 - 33s + 51$$

Now form the Routh table.

s^3	1	-33
s^2	-4	51
S^1	-20.25	
S^0	51	

There are two sign changes. Thus, there are two rhp poles and one lhp pole.

Chapter 7

7.1.

a. First check stability.

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{10s^2+500s+6000}{s^3+70s^2+1375s+6000}=\frac{10(s+30)(s+20)}{(s+26.03)(s+37.89)(s+6.085)}$$

Poles are in the lhp. Therefore, the system is stable. Stability also could be checked via Routh-Hurwitz using the denominator of $T(s)$. Thus,

$$15u(t):\ \ e_{step}(\infty)=\frac{15}{1+\lim\limits_{s\to 0}G(s)}=\frac{15}{1+\infty}=0$$

$$15tu(t):\ \ e_{ramp}(\infty)=\frac{15}{\lim\limits_{s\to 0}sG(s)}=\frac{15}{\frac{10*20*30}{25*35}}=2.1875$$

$$15t^2u(t):\ \ e_{parabola}(\infty)=\frac{15}{\lim\limits_{s\to 0}s^2G(s)}=\frac{30}{0}=\infty,\ \text{since}\ \boldsymbol{L}\ [15t^2]=\frac{30}{s^3}$$

b. First check stability.

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{10s^2+500s+6000}{s^5+110s^4+3875s^3+4.37e04s^2+500s+6000}$$

$$=\frac{10(s+30)(s+20)}{(s+50.01)(s+35)(s+25)(s^2-7.189e-04s+0.1372)}$$

From the second-order term in the denominator, we see that the system is unstable. Instability could also be determined using the Routh-Hurwitz criteria on the denominator of $T(s)$. Since the system is unstable, calculations about steady-state error cannot be made.

7.2.

a. The system is stable, since

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{1000(s+8)}{(s+9)(s+7)+1000(s+8)}=\frac{1000(s+8)}{s^2+1016s+8063}\ \text{and is of}$$

Type 0. Therefore,

$$K_p=\lim_{s\to 0}G(s)=\frac{1000*8}{7*9}=127;\ K_v=\lim_{s\to 0}sG(s)=0;\ \text{and}\ K_a=\lim_{s\to 0}s^2G(s)=0$$

b. $$e_{step}(\infty)=\frac{1}{1+\lim\limits_{s\to 0}G(s)}=\frac{1}{1+127}=7.8e-03$$

$$e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{0} = \infty$$

$$e_{parabola}(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{0} = \infty$$

7.3.

System is stable for positive *K*. System is Type 0. Therefore, for a step input $e_{step}(\infty) = \frac{1}{1+K_p} = 0.1$. Solving for K_p yields $K_p = 9 = \lim_{s\to 0} G(s) = \frac{12K}{14*18}$; from which we obtain $K = 189$.

7.4.

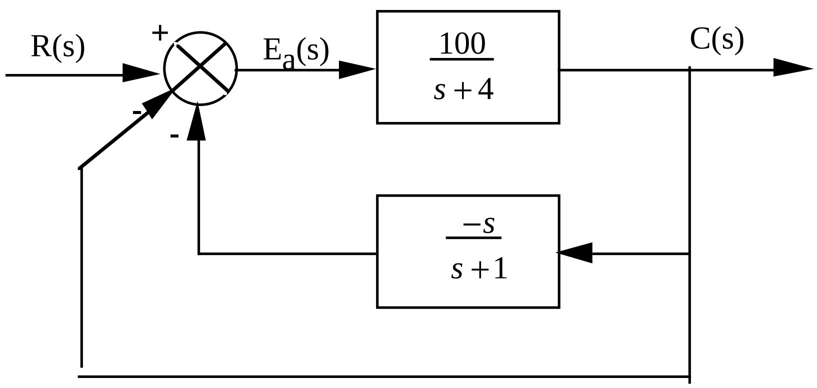
System is stable. Since $G_1(s) = 1000$, and $G_2(s) = \frac{(s+2)}{(s+4)}$,

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)} = -\frac{1}{2+1000} = -9.98e-04$$

7.5.

System is stable. Create a unity-feedback system, where $H_e(s) = \frac{1}{s+1} - 1 = \frac{-s}{s+1}$.

The system is as follows:



Thus,

$$G_e(s) = \frac{G(s)}{1+G(S)H_e(s)} = \frac{\frac{100}{(s+4)}}{1-\frac{100s}{(s+1)(s+4)}} = \frac{100(s+1)}{S^2-95s+4}$$

Hence, the system is Type 0. Evaluating K_p yields

$$K_p = \frac{100}{4} = 25$$

The steady-state error is given by

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+25} = 3.846e-02$$

7.6.

Since $G(s) = \dfrac{K(s+7)}{s^2+2s+10}$, $e(\infty) = \dfrac{1}{1+K_p} = \dfrac{1}{1+\dfrac{7K}{10}} = \dfrac{10}{10+7K}$.

Calculating the sensitivity, we get

$$S_{e:K} = \frac{K}{e}\frac{\partial e}{\partial K} = \frac{K}{\left(\dfrac{10}{10+7K}\right)}\frac{(-10)7}{(10+7K)^2} = -\frac{7K}{10+7K}$$

7.7.

Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix};\ \mathrm{R(s)} = \frac{1}{\mathrm{s}}.$$

Using the final value theorem,

$$e_{step}(\infty) = \lim_{s\to 0} sR(s)[1-\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}] = \lim_{s\to 0}[1-\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s & -1 \\ 3 & s+6 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$$

$$= \lim_{s\to 0}[1-\begin{bmatrix} 1 & 1 \end{bmatrix}\frac{\begin{bmatrix} s+6 & 1 \\ -3 & s \end{bmatrix}}{s^2+6s+3}\begin{bmatrix} 0 \\ 1 \end{bmatrix}] = \lim_{s\to 0}\frac{s^2+5s+2}{s^2+6s+3} = \frac{2}{3}$$

Using input substitution,

$$_{step}(\infty) = 1+\mathbf{CA}^{-1}\mathbf{B} = 1-\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 1+\begin{bmatrix} 1 & 1 \end{bmatrix}\frac{\begin{bmatrix} -6 & -1 \\ 3 & 0 \end{bmatrix}}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1+\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} = \frac{2}{3}$$

Chapter 8

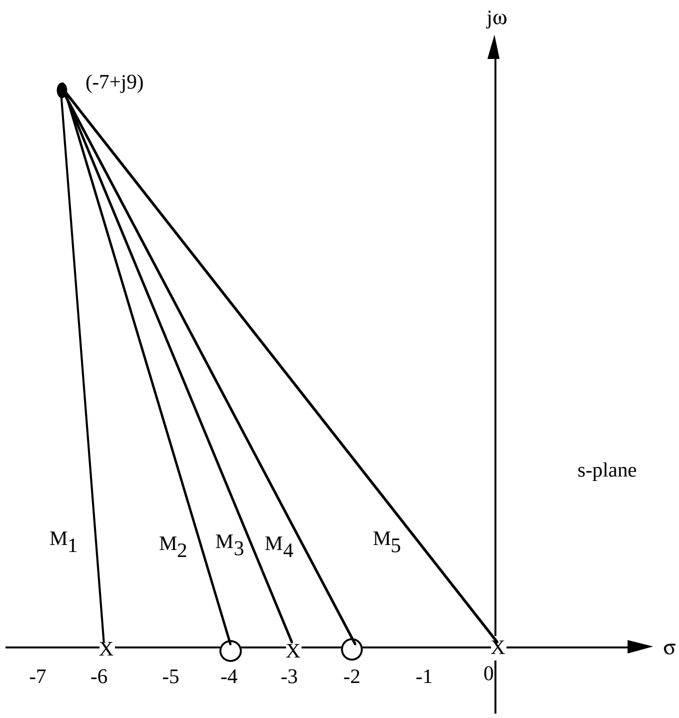
8.1.

a.

$$F(-7+j9)=\frac{(-7+j9+2)(-7+j9+4)0.0339}{(-7+j9)(-7+j9+3)(-7+j9+6)}=\frac{(-5+j9)(-3+j9)}{(-7+j9)(-4+j9)(-1+j9)}$$

$$=\frac{(-66-j72)}{(944-j378)}=-0.0339-j0.0899=0.096<-110.7^0$$

b. The arrangement of vectors is shown as follows:



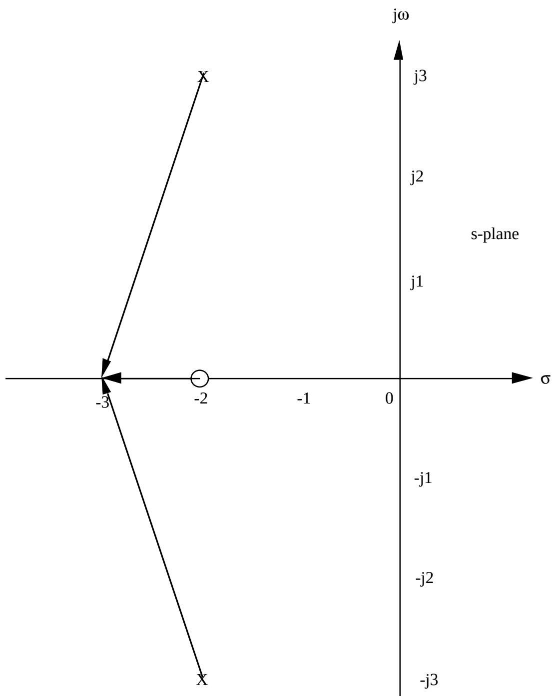
From the diagram,

$$F(-7+j9)=\frac{M_2M_4}{M_1M_3M_5}=\frac{(-3+j9)(-5+j9)}{(-1+j9)(-4+j9)(-7+j9)}$$

$$=\frac{(-66-j72)}{(944-j378)}=-0.0339-j0.0899=0.096<-110.7^0$$

8.2.

a. First draw the vectors.



From the diagram,

$$\sum \text{angles} = 180^0 - \tan^{-1}\left(\frac{-3}{-1}\right) - \tan^{-1}\left(\frac{-3}{1}\right) = 180^0 - 108.43^0 + 108.43^0 = 180^0.$$

b. Since the angle is 180^0, the point is on the root locus.

c. $K = \dfrac{\Pi \text{ pole lengths}}{\Pi \text{ zero lengths}} = \dfrac{\left(\sqrt{1^2+3^2}\right)\left(\sqrt{1^2+3^2}\right)}{1} = 10$

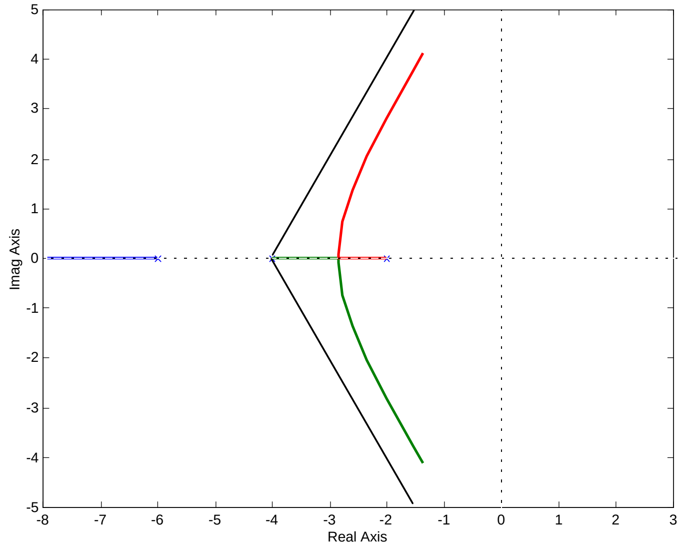
8.3.

First, find the asymptotes.

$$\sigma_a = \frac{\sum \text{poles - } \sum \text{zeros}}{\#\text{poles-}\#\text{zeros}} = \frac{(-2-4-6)-(0)}{3-0} = -4$$

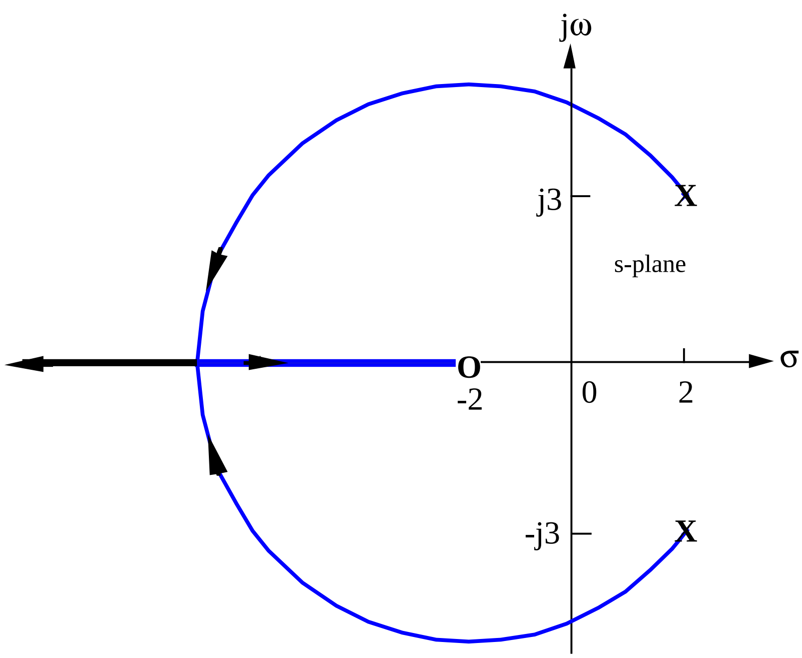
$$\theta_a = \frac{(2k+1)\pi}{3} = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$$

Next draw root locus following the rules for sketching.



8.4.

a.



b. Using the Routh-Hurwitz criteria, we first find the closed-loop transfer function. $T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{K(s+2)}{s^2+(K-4)s+(2K+13)}$

Using the denominator of T(s), make a Routh table.

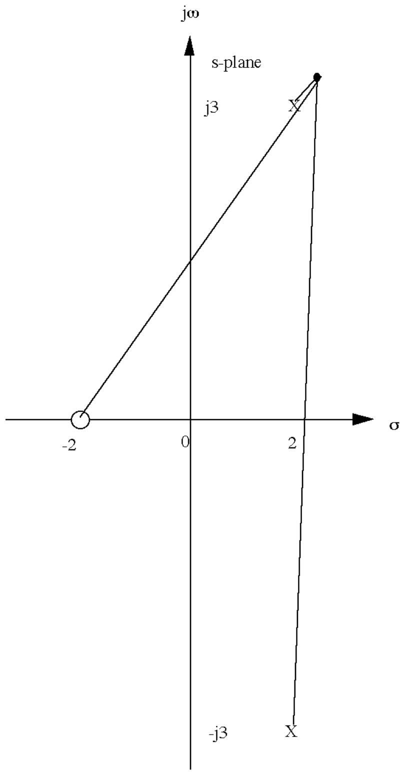
s^2	1	2*K*+13
s^1	*K*-4	0
s^0	2*K*+13	0

We get a row of zeros for $K = 4$. From the s^2 row with $K = 4$, $s^2 + 21 = 0$. From which we evaluate the imaginary axis crossing at $\sqrt{21}$.

c. From part (b), $K = 4$.

d. Searching for the minimum gain to the left of –2 on the real axis yields –7 at a gain of 18. Thus the break-in point is at –7.

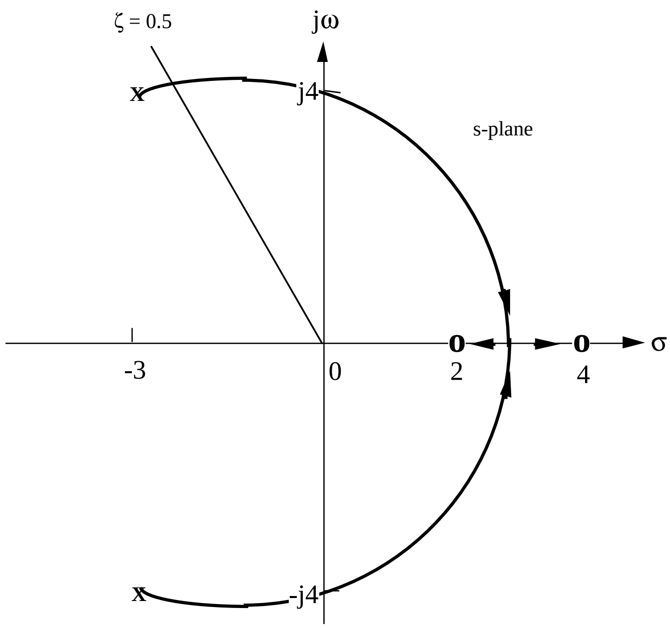
e. First, draw vectors to a point ε close to the complex pole.



At the point ε close to the complex pole, the angles must add up to zero. Hence, angle from zero – angle from pole in 4th quadrant – angle from pole in 1st quadrant $= 180^0$, or $\tan^{-1}\left(\frac{3}{4}\right) - 90^0 - \theta = 180^0$. Solving for the angle of departure, θ = -233.1.

8.5.

a.



b. Search along the imaginary axis and find the 180^0 point at $s = \pm j4.06$.

c. For the result in part (b), $K = 1$.

d. Searching between 2 and 4 on the real axis for the minimum gain yields the break-in at $s = 2.89$.

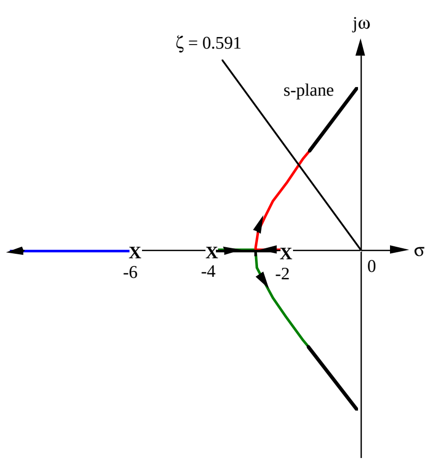
e. Searching along ζ = 0.5 for the 180^0 point we find $s = -2.42 + j4.18$.

f. For the result in part (e), $K = 0.108$.

g. Using the result from part (c) and the root locus, $K < 1$.

8.6.

a.



b. Searching along the ζ = 0.591 (10% overshoot) line for the 180^0 point yields - 2.028+j2.768 with K = 45.55.

c. $T_s = \frac{4}{|\text{Re}|} = \frac{4}{2.028} = 1.97$ s; $T_p = \frac{\pi}{|\text{Im}|} = \frac{\pi}{2.768} = 1.13$ s;

$\omega_n T_r = 1.8346$ from the rise-time chart and graph in Chapter 4. Since ω_n is the radial distance to the pole, $\omega_n = \sqrt{2.028^2 + 2.768^2} = 3.431$. Thus, $T_r = 0.53$ s; since the system is Type 0, $K_p = \frac{K}{2*4*6} = \frac{45.55}{48} = 0.949$. Thus,

$e_{step}(\infty) = \frac{1}{1+K_p} = 0.51$.

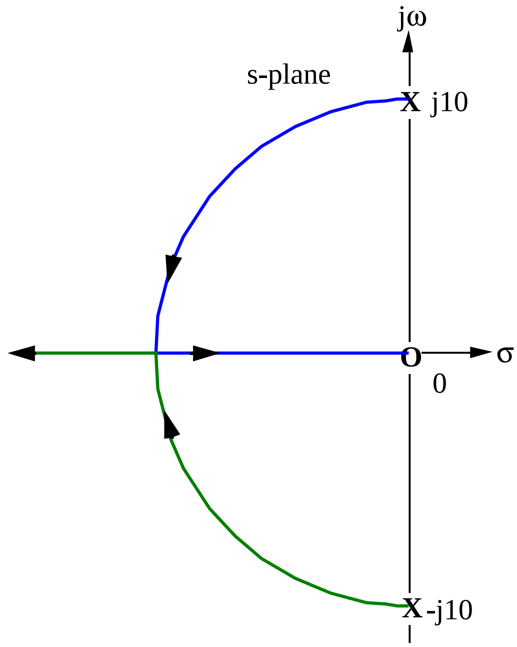
d. Searching the real axis to the left of –6 for the point whose gain is 45.55, we find –7.94. Comparing this value to the real part of the dominant pole, -2.028, we find that it is not five times further. The second-order approximation is not valid.

8.7.

Find the closed-loop transfer function and put it the form that yields p_i as the root locus variable. Thus,

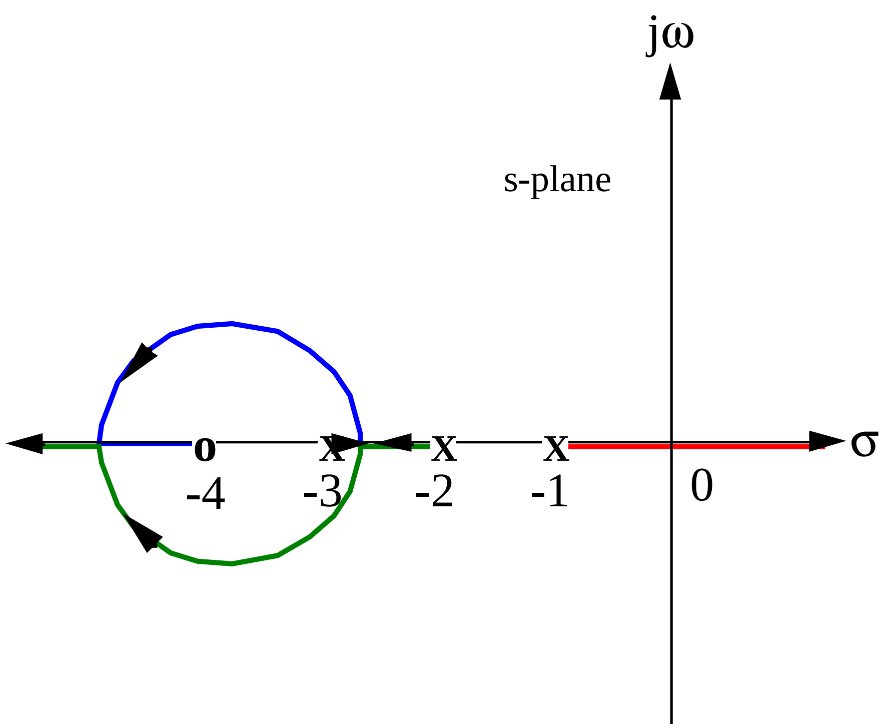
$$T(s) = \frac{G(s)}{1+G(s)} = \frac{100}{s^2 + p_i s + 100} = \frac{100}{(s^2+100) + p_i s} = \frac{\frac{100}{s^2+100}}{1 + \frac{p_i s}{s^2+100}}$$

Hence, $KG(s)H(s) = \frac{p_i s}{s^2+100}$. The following shows the root locus.



8.8.

Following the rules for plotting the root locus of positive-feedback systems, we obtain the following root locus:



8.9.

The closed-loop transfer function is $T(s) = \frac{K(s+1)}{s^2 + (K+2)s + K}$. Differentiating the denominator with respect to K yields

$$2s\frac{\partial s}{\partial K} + (K+2)\frac{\partial s}{\partial K} + (s+1) = (2s+K+2)\frac{\partial s}{\partial K} + (s+1) = 0$$

Solving for $\frac{\partial s}{\partial K}$, we get $\frac{\partial s}{\partial K} = \frac{-(s+1)}{(2s+K+2)}$. Thus, $S_{s:K} = \frac{K}{s}\frac{\partial s}{\partial K} = \frac{-K(s+1)}{s(2s+K+2)}$.

Substituting $K = 20$ yields $S_{s:K} = \frac{-10(s+1)}{s(s+11)}$.

Now find the closed-loop poles when $K = 20$. From the denominator of $T(s)$, $s_{1,2}$ = -21.05, - 0.95, -when $K = 20$.

For the pole at –21.05,

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = -21.05\left(\frac{-10(-21.05+1)}{-21.05(-21.05+11)}\right)0.05 = -0.9975.$$

For the pole at –0.95,

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = -0.95\left(\frac{-10(-0.95+1)}{-0.95(-0.95+11)}\right)0.05 = -0.0025.$$

Chapter 9

9.1.

a. Searching along the 15% overshoot line, we find the point on the root locus at –3.5 + j5.8 at a gain of K = 45.84. Thus, for the uncompensated system, $K_v = \lim_{s\to 0} sG(s) = K/7 = 45.84/7 = 6.55$.

Hence, $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1527$.

b. Compensator zero should be 20x further to the left than the compensator pole.

Arbitrarily select $G_c(s) = \dfrac{(s+0.2)}{(s+0.01)}$.

c. Insert compensator and search along the 15% overshoot line and find the root locus at

–3.4 + j5.63 with a gain, K = 44.64. Thus, for the compensated

system, $K_v = \dfrac{44.64(0.2)}{(7)(0.01)} = 127.5$ and $e_{ramp\_compensated}(\infty) = \dfrac{1}{K_v} = 0.0078$.

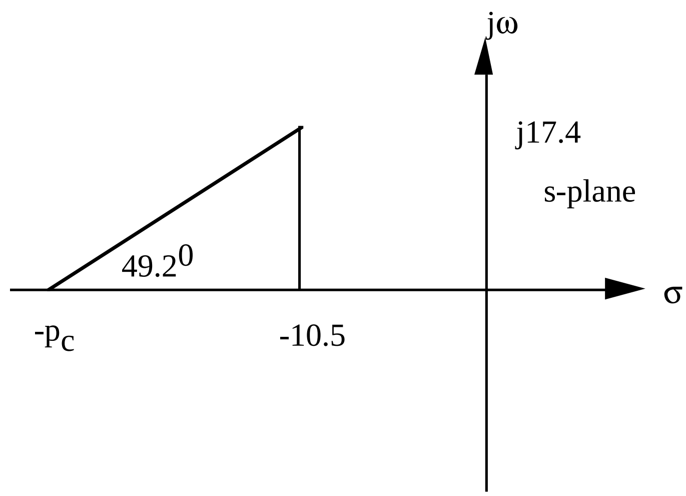
d. $\dfrac{e_{ramp\_uncompensated}}{e_{ramp\_compensated}} = \dfrac{0.1527}{0.0078} = 19.58$

9.2.

a. Searching along the 15% overshoot line, we find the point on the root locus at –3.5 + j5.8 at a gain of K = 45.84. Thus, for the uncompensated system,

$T_s = \dfrac{4}{|\text{Re}|} = \dfrac{4}{3.5} = 1.143$ s.

b. The real part of the design point must be three times larger than the uncompensated pole's real part. Thus the design point is 3(-3.5) + j 3(5.8) = -10.5 + j17.4. The angular contribution of the plant's poles and compensator zero at the design point is 130.8^0. Thus, the compensator pole must contribute $180^0 – 130.8^0 = 49.2^0$. Using the following diagram,



we find $\frac{17.4}{p_c - 10.5} = \tan 49.2^0$, from which, p_c = 25.52. Adding this pole, we find the gain at the design point to be K = 476.3. A higher-order closed-loop pole is found to be at –11.54. This pole may not be close enough to the closed-loop zero at –10. Thus, we should simulate the system to be sure the design requirements have been met.

9.3.

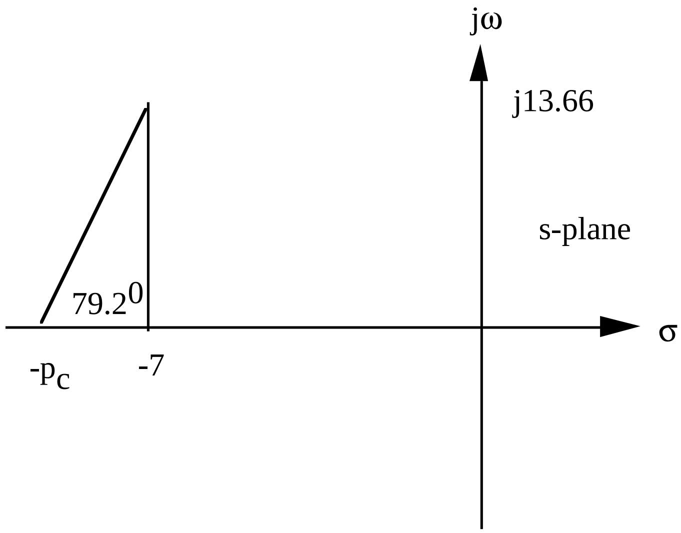
a. Searching along the 20% overshoot line, we find the point on the root locus at –3.5 + 6.83 at a gain of K = 58.9. Thus, for the uncompensated system,

$$T_s = \frac{4}{|\text{Re}|} = \frac{4}{3.5} = 1.143 \text{ s}.$$

b. For the uncompensated system, $K_v = \lim_{s\to 0} sG(s) = K/7 = 58.9/7 = 8.41$. Hence,

$$e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1189.$$

c. In order to decrease the settling time by a factor of 2, the design point is twice the uncompensated value, or –7 + j13.66. Adding the angles from the plant's poles and the compensator's zero at –3 to the design point, we obtain -100.8^0. Thus, the compensator pole must contribute $180^0 - 100.8^0 = 79.2^0$. Using the following diagram,



we find $\frac{13.66}{p_c - 7} = \tan 79.2^0$, from which, p_c = 9.61. Adding this pole, we find the gain at the design point to be K = 204.9.

Evaluating K_v for the lead-compensated system:

$$K_v = \lim_{s\to 0} sG(s)G_{lead} = K(3)/[(7)(9.61)] = (204.9)(3)/[(7)(9.61)] = 9.138.$$

K_v for the uncompensated system was 8.41. For a 10x improvement in steady-state error, K_v must be (8.41)(10) = 84.1. Since lead compensation gave us K_v = 9.138, we need an improvement of 84.1/9.138 = 9.2.

Thus, the lag compensator zero should be 9.2x further to the left than the compensator pole. Arbitrarily select $G_c(s) = \frac{(s+0.092)}{(s+0.01)}$.

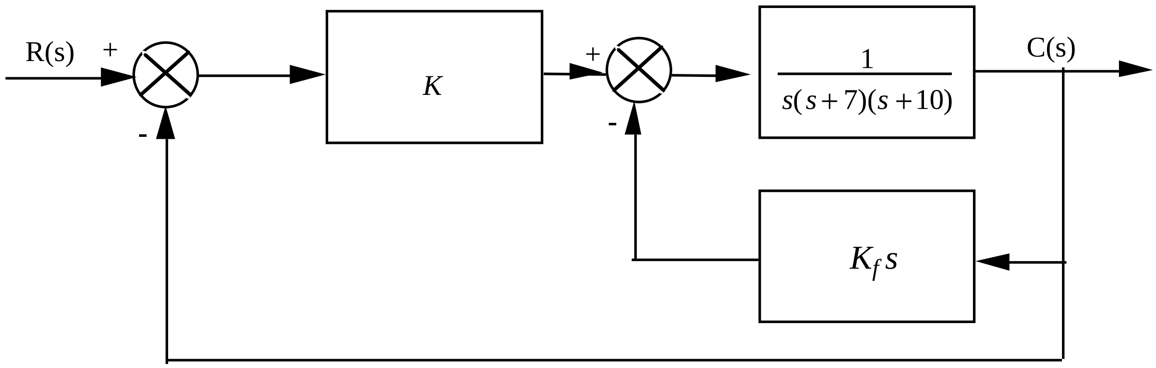
Using all plant and compensator poles, we find the gain at the design point to be K = 205.4. Summarizing the forward path with plant, compensator, and gain yields

$$G_e(s) = \frac{205.4(s+3)(s+0.092)}{s(s+7)(9.61)(s+0.01)}.$$

Higher-order poles are found at –0.928 and –2.6. It would be advisable to simulate the system to see if there is indeed pole-zero cancellation.

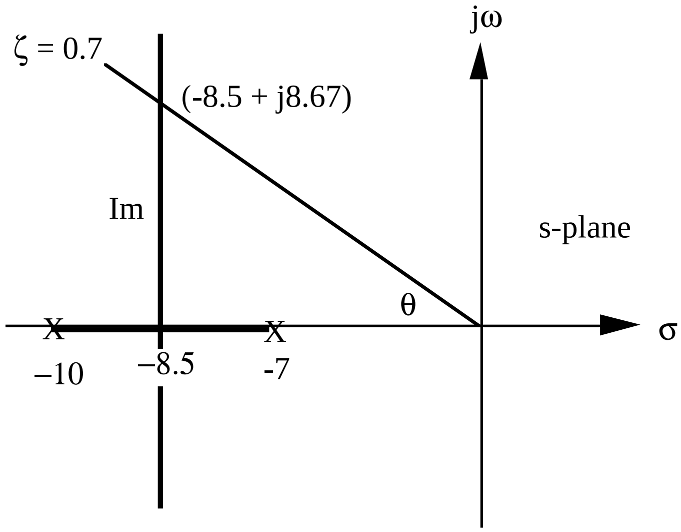
9.4.

The configuration for the system is shown in the figure below.



Minor-Loop Design:

For the minor loop, $G(s)H(s) = \dfrac{K_f}{(s+7)(s+10)}$. Using the following diagram, we find that the minor-loop root locus intersects the 0.7 damping ratio line at –8.5 + j8.67. The imaginary part was found as follows: $\theta = \cos^{-1}\zeta = 45.57^0$. Hence, $\dfrac{\text{Im}}{8.5} = \tan 45.57^0$, from which Im = 8.67.



The gain, K_f, is found from the vector lengths as

$$K_f = \sqrt{1.5^2 + 8.67^2}\sqrt{1.5^2 + 8.67^2} = 77.42$$

Major-Loop Design:

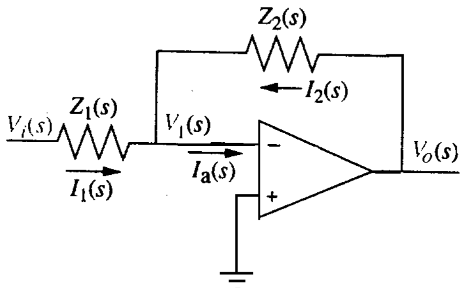
Using the closed-loop poles of the minor loop, we have an equivalent forward-path transfer function of

$$G_e(s) = \frac{K}{s(s+8.5+j8.67)(s+8.5-j8.67)} = \frac{K}{s(s^2+17s+147.4)}.$$

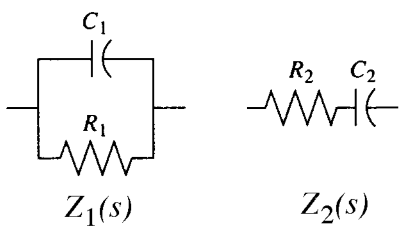
Using the three poles of $G_e(s)$ as open-loop poles to plot a root locus, we search along ζ = 0.5 and find that the root locus intersects this damping ratio line at –4.34 + j7.51 at a gain, $K = 626.3$.

9.5.

a. An active PID controller must be used. We use the circuit shown in the following figure:



where the impedances are shown below as follows:



Matching the given transfer function with the transfer function of the PID controller yields

$$G_c(s) = \frac{(s+0.1)(s+5)}{s} = \frac{s^2+5.1s+0.5}{s} = s+5.1+\frac{0.5}{s} = -\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$$

Equating coefficients

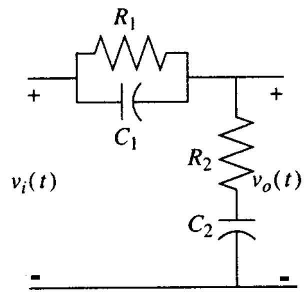
$$\frac{1}{R_1C_2} = 0.5 \qquad (1)$$

$$R_2C_1 = 1 \qquad (2)$$

$$\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right) = 5.1 \quad (3)$$

In Eq. (2) we arbitrarily let $C_1 = 10^{-5}$. Thus, $R_2 = 10^5$. Using these values along with Eqs. (1) and (3) we find $C_2 = 100\ \mu\text{F}$ and $R_1 = 20\ \text{k}\Omega$.

b. The lag-lead compensator can be implemented with the following passive network, since the ratio of the lead pole-to-zero is the inverse of the ratio of the lag pole-to-zero:



Matching the given transfer function with the transfer function of the passive lag-lead compensator yields

$$G_c(s) = \frac{(s+0.1)(s+2)}{(s+0.01)(s+20)} = \frac{(s+0.1)(s+2)}{s^2+20.01s+0.2} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$$

Equating coefficients

$$\frac{1}{R_1C_1} = 0.1 \qquad (1)$$

$$\frac{1}{R_2C_2} = 2 \qquad (2)$$

$$\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right) = 20.01 \qquad (3)$$

Substituting Eqs. (1) and (2) in Eq. (3) yields

$$\frac{1}{R_2C_1} = 17.91 \qquad (4)$$

Arbitrarily letting $C_1 = 100\ \mu\text{F}$ in Eq. (1) yields $R_1 = 100\ \text{k}\Omega$.

Substituting $C_1 = 100\ \mu\text{F}$ into Eq. (4) yields $R_2 = 558\ \text{k}\Omega$.

Substituting $R_2 = 558\ \text{k}\Omega$ into Eq. (2) yields $C_2 = 900\ \mu\text{F}$.

Chapter 10

10.1.

a.

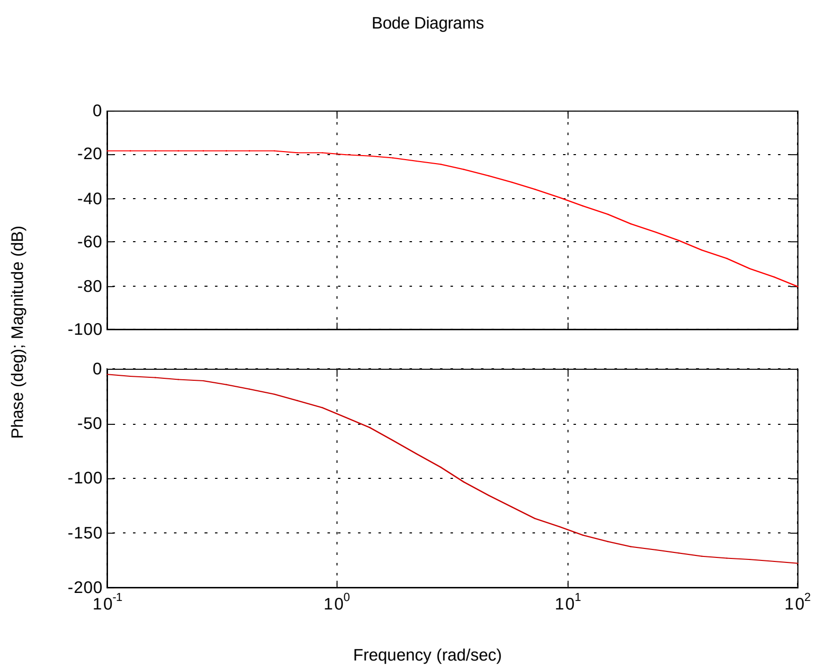
$$G(s)=\frac{1}{(s+2)(s+4)};\ \mathrm{G}(\mathrm{j}\omega)=\frac{1}{(8-\omega^2)+j6\omega}$$

$$M(\omega)=\sqrt{(8-\omega^2)^2+(6\omega)^2}$$

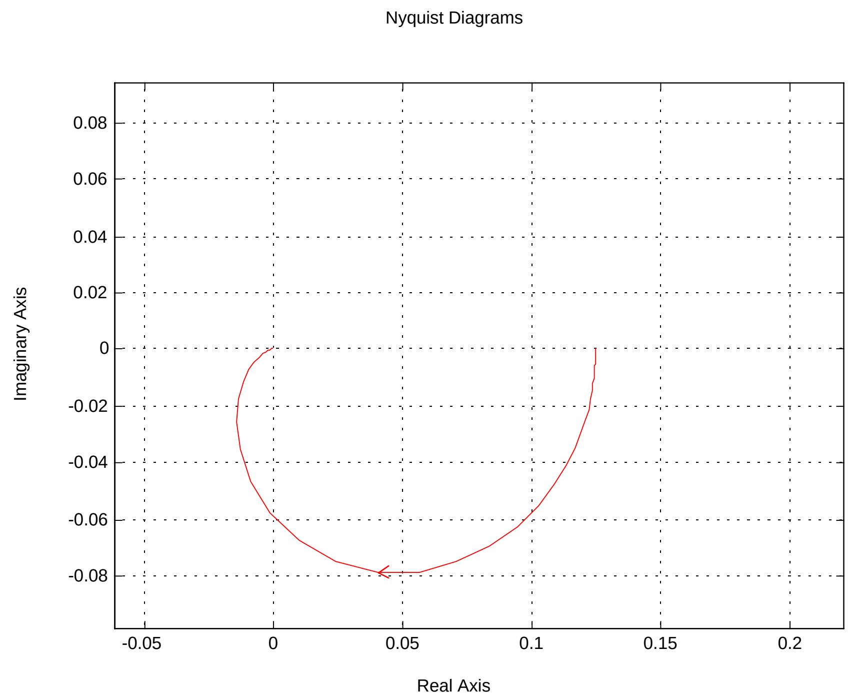
For $\omega<\sqrt{8},\ \phi(\omega)=-\tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right)$.

For $\omega>\sqrt{8},\ \phi(\omega)=-\left(\pi+\tan^{-1}\left[\frac{6\omega}{8-\omega^2}\right]\right)$.

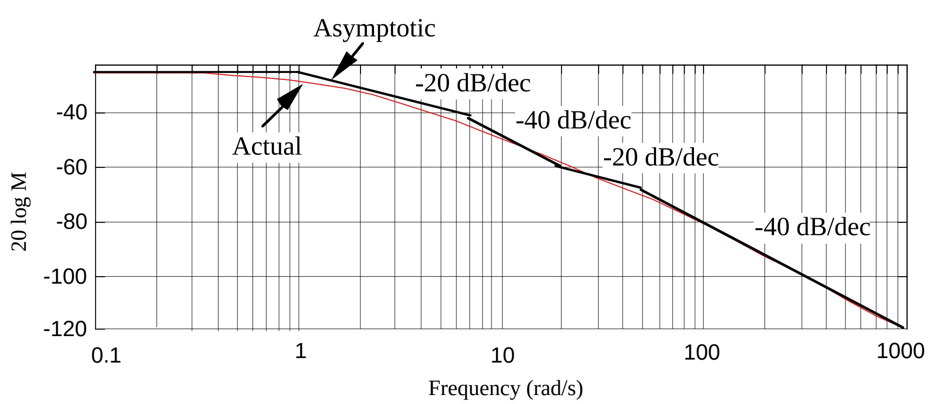
b.

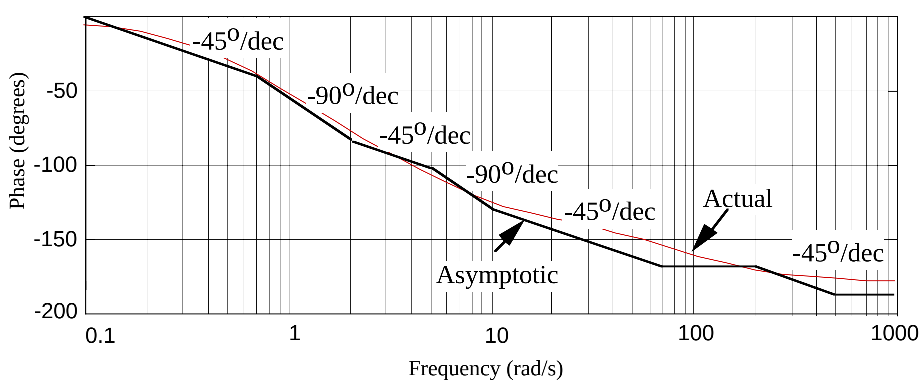


c.


Nyquist Diagrams
Imaginary Axis
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.05
0
0.05
0.1
0.15
0.2
Real Axis

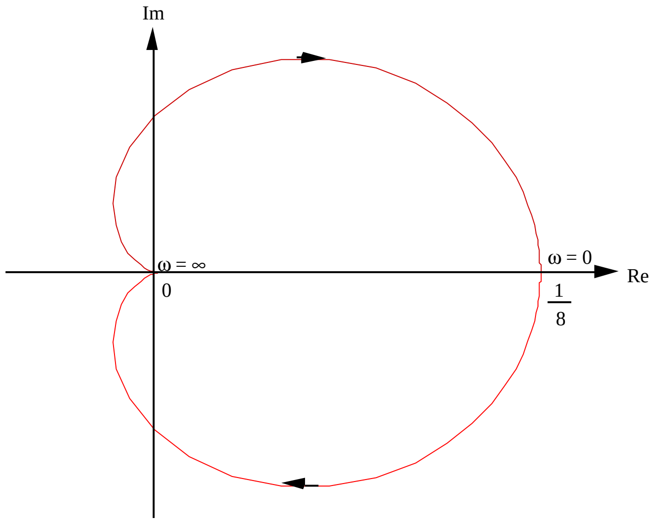
10.2.


Asymptotic
Actual
-20 dB/dec
-40 dB/dec
-20 dB/dec
-40 dB/dec
20 log M
-40
-60
-80
-100
-120
0.1
1
10
100
1000
Frequency (rad/s)


-45°/dec
-90°/dec
-45°/dec
-90°/dec
-45°/dec
Actual
-45°/dec
Asymptotic
Phase (degrees)
-50
-100
-150
-200
0.1
1
10
100
1000
Frequency (rad/s)

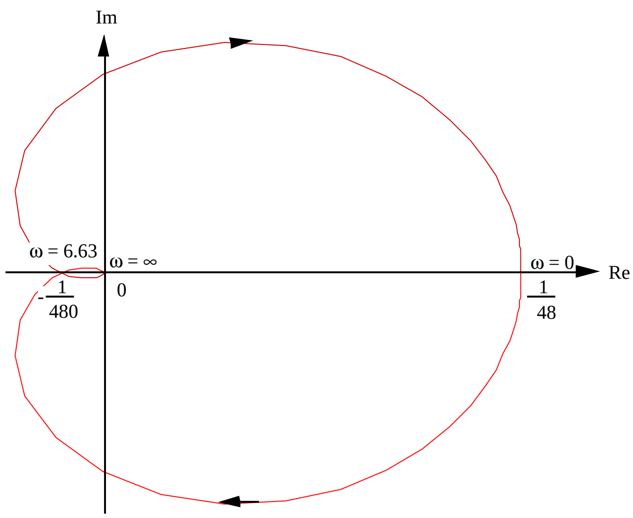
10.3.

The frequency response is 1/8 at an angle of zero degrees at $\omega = 0$. Each pole rotates 90^0 in going from $\omega = 0$ to $\omega = \infty$. Thus, the resultant rotates -180^0 while its magnitude goes to zero. The result is shown below.



10.4.

a. The frequency response is 1/48 at an angle of zero degrees at $\omega = 0$. Each pole rotates 90^0 in going from $\omega = 0$ to $\omega = \infty$. Thus, the resultant rotates -270^0 while its magnitude goes to zero. The result is shown below.



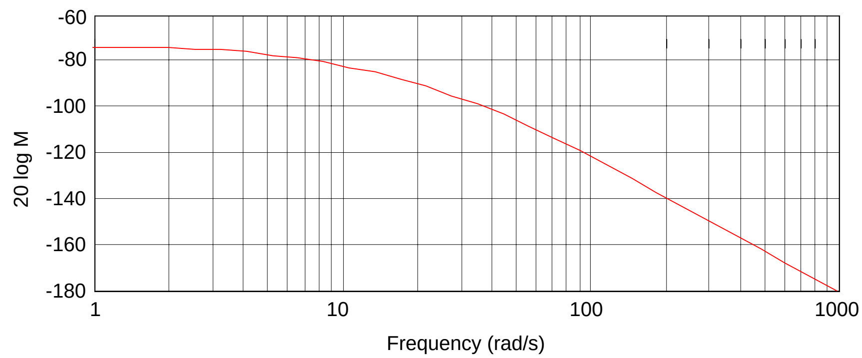
b. Substituting $j\omega$ into $G(s) = \dfrac{1}{(s+2)(s+4)(s+6)} = \dfrac{1}{s^3 + 12s^2 + 44s + 48}$ and simplifying, we obtain $G(j\omega) = \dfrac{(48 - 12\omega^2) - j(44\omega - \omega^3)}{\omega^6 + 56\omega^4 + 784\omega^2 + 2304}$. The Nyquist diagram crosses the real axis when the imaginary part of $G(j\omega)$ is zero. Thus, the Nyquist diagram crosses the real axis at $\omega^2 = 44$, or $\omega = \sqrt{44} = 6.63$ rad/s. At this frequency $G(j\omega) = -\frac{1}{480}$. Thus, the system is stable for $K < 480$.

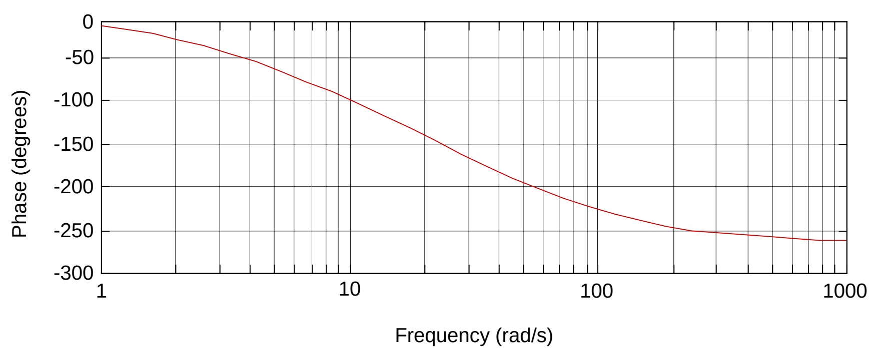
10.5.

If K = 100, the Nyquist diagram will intersect the real axis at –100/480. Thus, $G_M = 20\log\frac{480}{100} = 13.62$ dB. From Skill-Assessment Exercise Solution 10.4, the 180^0 frequency is 6.63 rad/s.

10.6.

a.





b. The phase angle is 180^0 at a frequency of 36.74 rad/s. At this frequency the gain is –99.67 dB. Therefore, $20\log K = 99.67$, or $K = 96,270$. We conclude that the system is stable for $K < 96,270$.

c. For $K = 10,000$, the magnitude plot is moved up $20\log 10,000 = 80$ dB. Therefore, the gain margin is 99.67- 80 = 19.67 dB. The 180^0 frequency is 36.7

rad/s. The gain curve crosses 0 dB at $\omega = 7.74$ rad/s, where the phase is 87.1^0. We calculate the phase margin to be $180^0 - 87.1^0 = 92.9^0$.

10.7.

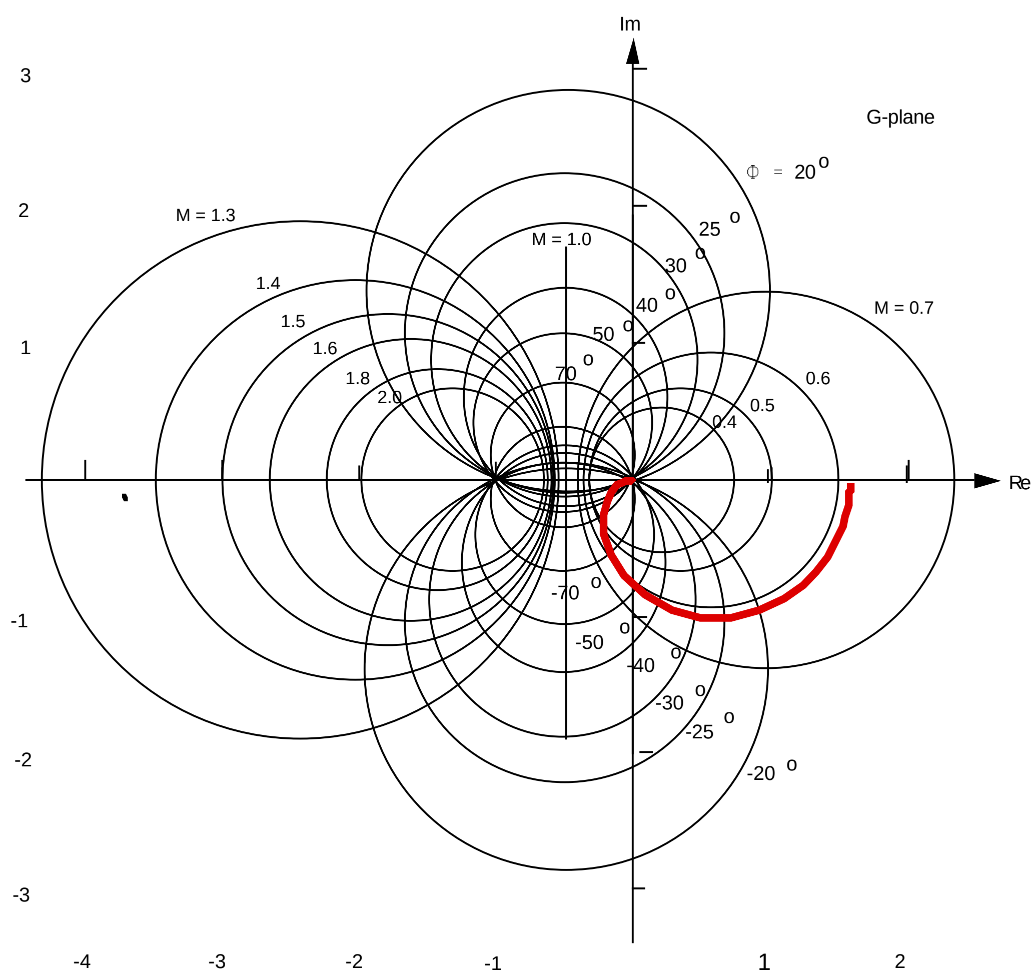
Using $\zeta = \frac{-\ln(\%/100)}{\sqrt{\pi^2 + \ln^2(\%/100)}}$, we find $\zeta = 0.456$, which corresponds to 20% overshoot. Using $T_s = 2$, $\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 5.79$ rad/s.

10.8.

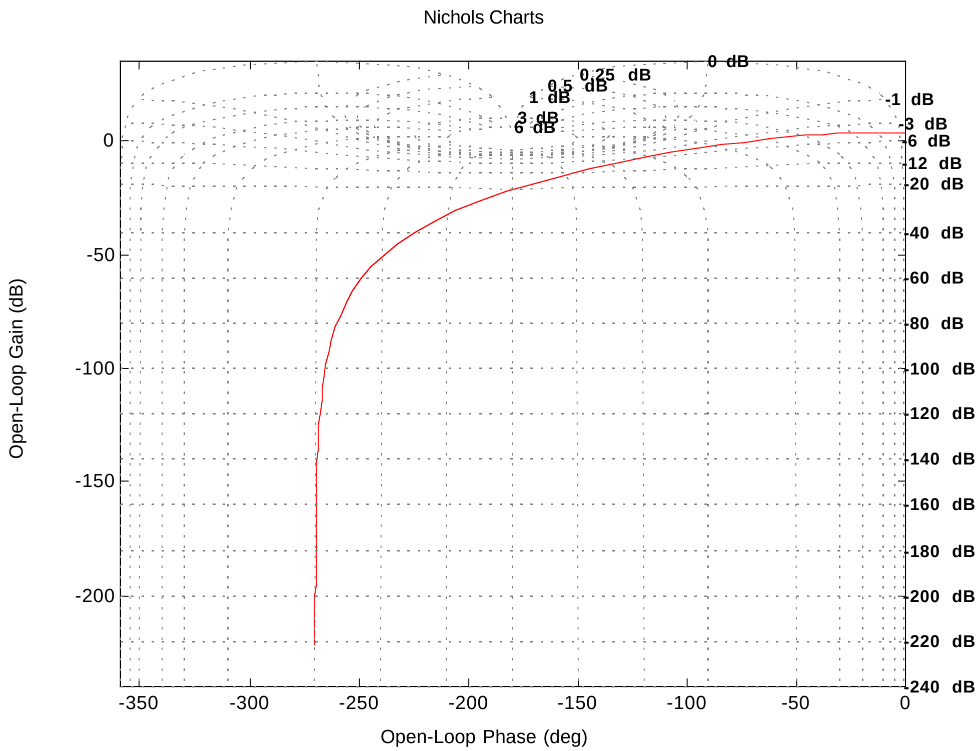
For both parts find that

$G(j\omega) = \frac{160}{27} * \frac{(6750000 - 101250\omega^2) + j1350(\omega^2 - 1350)\omega}{\omega^6 + 2925\omega^4 + 1072500\omega^2 + 25000000}$. For a range of values for ω, superimpose $G(j\omega)$ on the **a.** M and N circles, and on the **b.** Nichols chart.

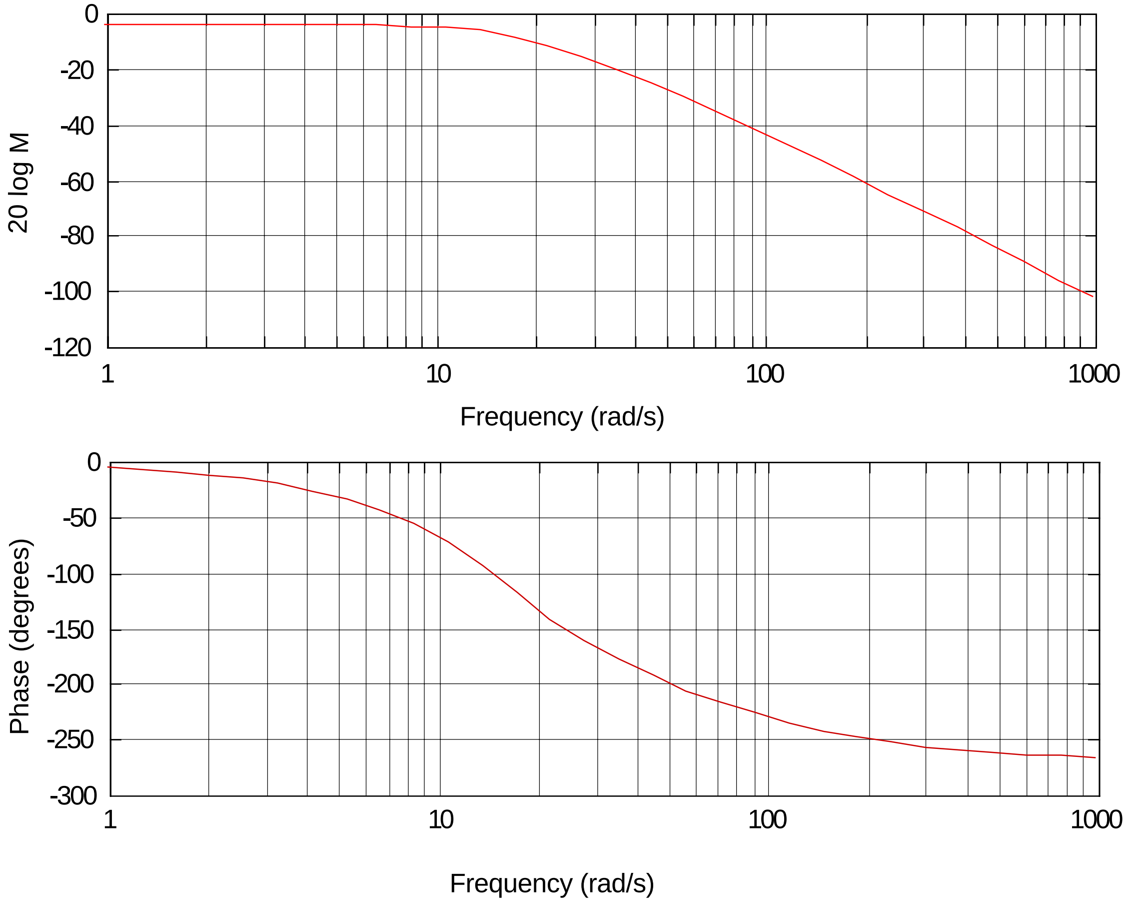
a.



b.

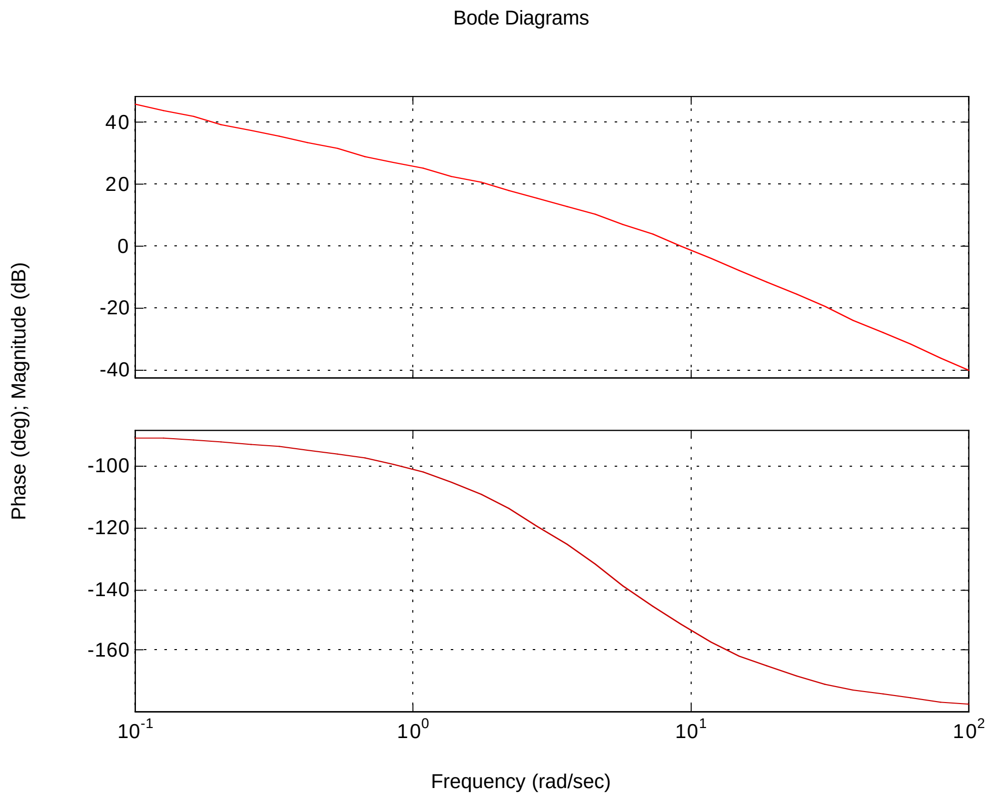


Plotting the closed-loop frequency response from **a.** or **b.** yields the following plot:



10.9.

The open-loop frequency response is shown in the following figure:



The open-loop frequency response is –7 at $\omega = 14.5$ rad/s. Thus, the estimated bandwidth is $\omega_{WB} = 14.5$ rad/s. The open-loop frequency response plot goes through zero dB at a frequency of 9.4 rad/s, where the phase is 151.98^0. Hence, the phase margin is $180^0 - 151.98^0 = 28.02^0$. This phase margin corresponds to $\zeta = 0.25$. Therefore, $\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} x100 = 44.4\%$,

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} = 1.64 \text{ s and}$$

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} = 0.33 \text{ s}$$

10.10.

The initial slope is 40 dB/dec. Therefore, the system is Type 2. The initial slope intersects 0 dB at $\omega = 9.5$ rad/s. Thus, $K_a = 9.5^2 = 90.25$ and $K_p = K_v = \infty$.

10.11.

a. Without delay, $G(j\omega) = \frac{10}{j\omega(j\omega+1)} = \frac{10}{\omega(-\omega+j)}$, from which the zero dB

frequency is found as follows: $M = \frac{10}{\omega\sqrt{\omega^2+1}} = 1$. Solving for ω,

$\omega\sqrt{\omega^2+1} = 10$, or after squaring both sides and rearranging, $\omega^4 + \omega^2 - 100 = 0$.

Solving for the roots, $\omega^2 = -10.51,\ 9.51$. Taking the square root of the positive root, we find the 0 dB frequency to be 3.08 rad/s. At this frequency, the phase angle, $\phi = -\angle(-\omega + j) = -\angle(-3.08 + j) = -162^{\circ}$. Therefore the phase margin is $180^0 - 162^0 = 18^0$.

b. With a delay of 3 s,

$\phi = -\angle(-\omega + j) - \omega T = -\angle(-3.08 + j) - (3.08)(3) \quad = -162^{\circ} - 9.24^{\circ} = -171.24^{\circ}$.

Therefore the phase margin is $180^0 - 171.24^0 = 8.76^0$.

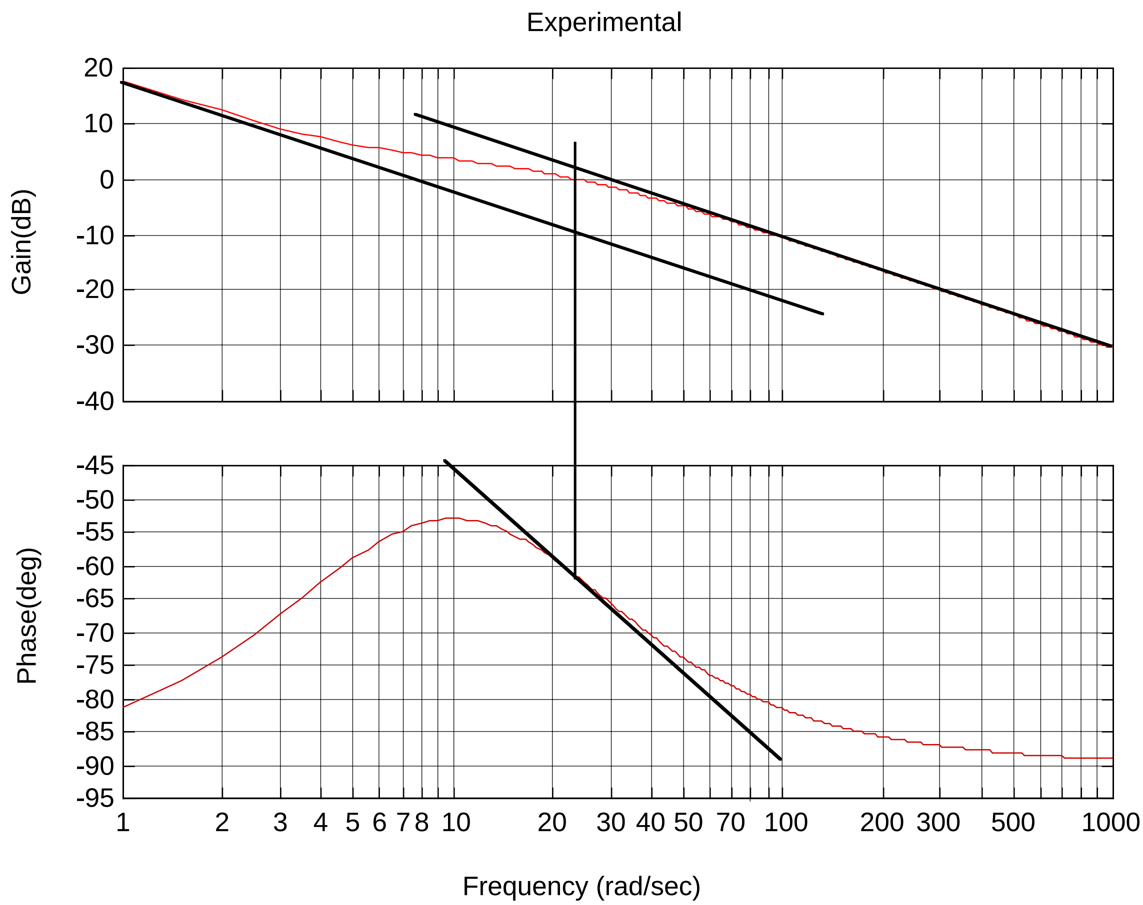
c. With a delay of 7 s,

$\phi = -\angle(-\omega + j) - \omega T = -\angle(-3.08 + j) - (3.08)(7) = -162^{\circ} - 21.56^{\circ} = -183.56^{\circ}$.

Therefore the phase margin is $180^0 - 183.56^0 = -3.56^0$. Thus, the system is unstable.

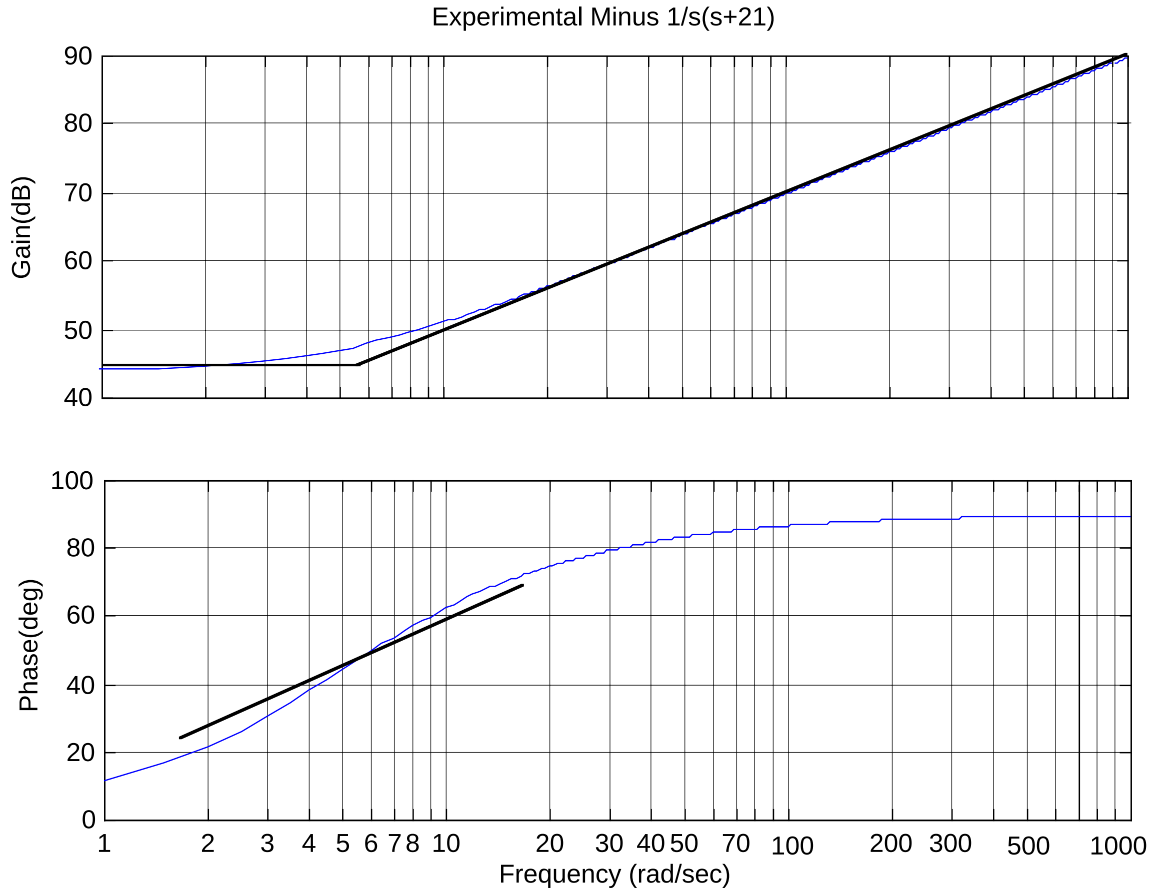
10.12.

Drawing judicially selected slopes on the magnitude and phase plot as shown below yields a first estimate.



We see an initial slope on the magnitude plot of –20 dB/dec. We also see a final –20 dB/dec slope with a break frequency around 21 rad/s. Thus, an initial estimate is $G_1(s) = \frac{1}{s(s+21)}$.

Subtracting $G_1(s)$ from the original frequency response yields the frequency response shown below.



Drawing judicially selected slopes on the magnitude and phase plot as shown yields a final estimate. We see first-order zero behavior on the magnitude and phase plots with a break frequency of about 5.7 rad/s and a dc gain of about 44 dB $= 20\log(5.7K)$, or $K = 27.8$. Thus, we estimate $G_2(s) = 27.8(s+7)$. Thus,

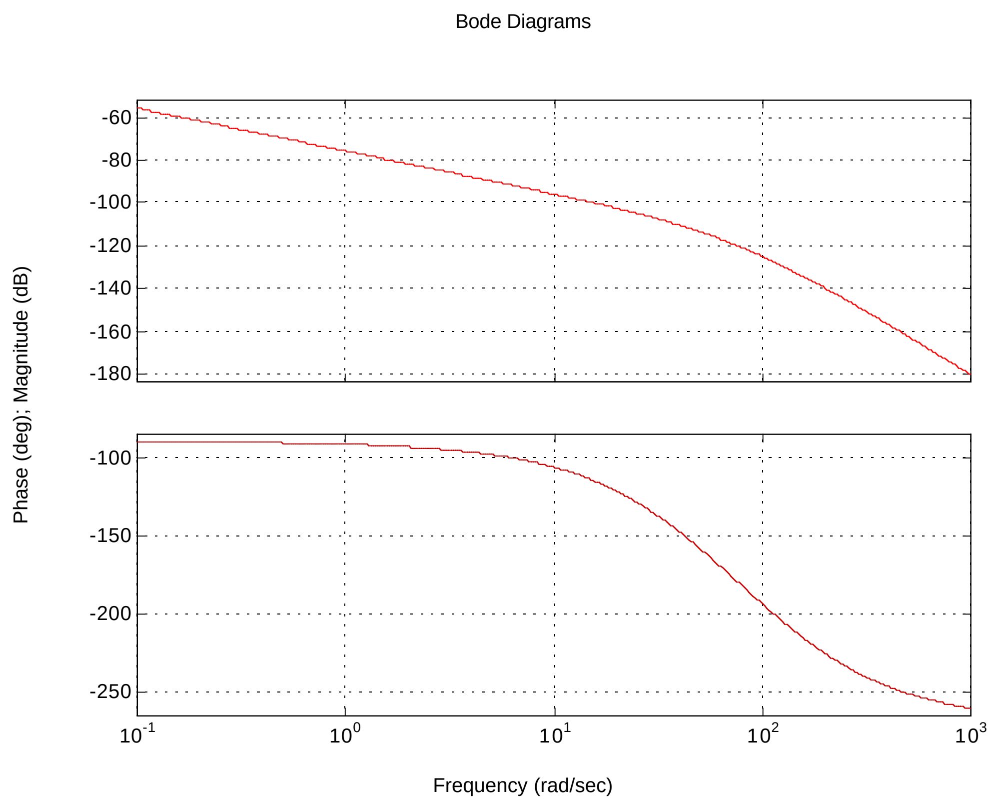
$$G(s) = G_1(s)G_2(s) = \frac{27.8(s+5.7)}{s(s+21)}$$. It is interesting to note that the original

problem was developed from $G(s) = \dfrac{30(s+5)}{s(s+20)}$.

Chapter 11

11.1.

The Bode plot for K = 1 is shown below.



A 20% overshoot requires $\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$. This damping ratio

implies a phase margin of 48.10, which is obtained when the _ = -1800 + 48.10 = 131.9^0. This phase angle occurs at $\omega = 27.6$ rad/s. The magnitude at this frequency is 5.15 x 10^{-6}. Since the magnitude must be

unity $K = \frac{1}{5.15x10^{-6}} = 194,200$.