CHAPTER 5

CHEBYSHEV TYPE II FILTERS

Chebyshev Type II filters are closely related to Chebyshev Type I filters, and are noted for having a flat passband magnitude response, and an equiripple response in the stopband. As was noted in Chapter 4, the Chebyshev Type I response is often simply referred to as the Chebyshev response. Similarly, the Chebyshev Type II response is often referred to as the Inverse Chebyshev response, for reasons that will become clear as the response is developed below.

In this chapter, the Chebyshev Type II response is defined, and it will be observed that it satisfies the Analog Filter Design Theorem. Explicit formulas for the design and analysis of Chebyshev Type II filters, such as Filter Selectivity, Shaping Factor, the minimum required order to meet design specifications, etc., will be obtained. From the defining $|H(j\omega)|^2$ the corresponding $H(s)$ will be determined, and means for determining the filter poles and zeros are found. To complete the study of lowpass, prototype Chebyshev Type II filters, the phase response, phase delay, group delay, and time-domain response characteristics are investigated.

5.1 EQUIRIPPLE STOPBAND MAGNITUDE

Suppose that $|G(j\omega)|^2$ is the magnitude-squared frequency response of a Chebyshev Type I filter according to (4.1):

$$|G(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 C_N^2(\omega/\omega_p)}.$$  

Let $|F(j\omega)|^2$ be obtained from (5.1) as follows:

$$|F(j\omega)|^2 = 1 - |G(j\omega)|^2 = \frac{\varepsilon^2 C_N^2(\omega/\omega_p)}{1 + \varepsilon^2 C_N^2(\omega/\omega_p)}.$$  

Finally, the desired magnitude-squared response is obtained by replacing $\omega/\omega_p$ by $\omega_s/\omega$ in (5.2):
Definition of the magnitude-squared Chebyshev Type II response:

\[
|H(j\omega)|^2 = \frac{\varepsilon^2 C_N^2(\omega_s/\omega)}{1 + \varepsilon^2 C_N^2(\omega_s/\omega)}. \tag{5.3}
\]

where

\[
C_N(\omega_s/\omega) = \begin{cases} 
\cos[N \cos^{-1}(\omega_s/\omega)], & |\omega| \geq \omega_s \\
\cosh[N \cosh^{-1}(\omega_s/\omega)], & |\omega| \leq \omega_s
\end{cases} \tag{5.4}
\]

and \(\omega_s\) is a frequency scaling constant, and \(\varepsilon\) is a constant that adjusts the influence of \(C_N(\omega_s/\omega)\) in the denominator of \(|H(j\omega)|^2\). Therefore, it is observed that the hyperbolic cosine is used in (5.4) for low frequencies, and, from (5.3) that this results in a response near unity; the trigonometric cosine is used for high frequencies beyond \(\omega_s\) resulting in a rippling response of small magnitude.

In due course it will be shown that (5.4) can be expressed as a polynomial, in fact very closely related to the Chebyshev polynomials of Section 4.4, and that as such (5.3) will satisfy the Analog Filter Design Theorem, and therefore the imposed constraints of Section 2.6 will be satisfied. It will be shown that \(N\) is the order of the Chebyshev polynomial, and in Section 5.5 it will be shown that \(N\) is the order of the filter, i.e., the number of poles of the transfer function \(H(s)\). The form shown for \(C_N(\omega_s/\omega)\) in (5.4) is very convenient for analytical investigation purposes, revealing the characteristics of the Chebyshev Type II response, and also yielding design formulae such as for the minimum required order to meet design specifications.

Note that \(0 \leq C_N^2(\omega_s/\omega) \leq 1\), for \(\omega \geq \omega_s\), and \(C_N^2(\omega_s/\omega) \leq 1\), for \(0 \leq \omega \leq \omega_s\). Therefore, \(\omega \geq \omega_s\) defines the stopband, and \(|H(j\omega)|^2\) ripples within the stopband following the cosine function. Within the passband, as can be seen from (5.3) and (5.4), the magnitude-squared frequency response follows the hyperbolic cosine function and falls off monotonically for increasing \(\omega\).

It is easy to see that

\[
|H(j\omega)|^2|_{\omega=0} = 1,
\]

independent of \(N\), and that

\[
|H(j\omega)|^2|_{\omega=\omega_s} = \frac{\varepsilon^2}{1 + \varepsilon^2}.
\]

In terms of \(dB\),
10 \log |H(j\omega)|_{\omega=0} = 0 ,

and

10 \log |H(j\omega)|_{\omega=\omega_s} = 10 \log \left[ \epsilon^2/(1 + \epsilon^2) \right] . \tag{5.5}

Note that (5.5) is the minimum attenuation for all \( \omega \geq \omega_s \). When (5.5) is compared with the general magnitude specifications for the design of a lowpass filter illustrated in Figure 2.15 on page 52, setting \( A_s \) equal to the negative of (5.5) results in

\[
\epsilon = \frac{1}{\sqrt{10^4/A_s^2 - 1}} . \tag{5.6}
\]

Several values of \( A_s \) and corresponding values of \( \epsilon \) are shown in Table 5.1. Note that \( A_s \) is the minimum attenuation in the stopband. At frequencies where the numerator of (5.3) is zero, the attenuation is infinity.

Note that the magnitude-squared response of (5.3) is zero in the stopband when \( C_N^2(\omega_s/\omega) = 0 \). The frequencies where the response is zero may be found as follows:

\[
C_N^2(\omega_s/\omega) = \cos^2[N \cos^{-1}(\omega_s/\omega)] = 0 ,
\]

from which

\[
N \cos^{-1}(\omega_s/\omega) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots, \frac{k\pi}{2}, \quad k = 1, 3, 5, \cdots .
\]

### Table 5.1

<table>
<thead>
<tr>
<th>( A_s ), dB</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.1005038</td>
</tr>
<tr>
<td>30</td>
<td>0.0316386</td>
</tr>
<tr>
<td>40</td>
<td>0.0100005</td>
</tr>
<tr>
<td>50</td>
<td>0.0031623</td>
</tr>
<tr>
<td>60</td>
<td>0.0010000</td>
</tr>
</tbody>
</table>

Section 5.1  
Equiripple Stopband Magnitude
Therefore, the frequencies where the response is zero are as follows:

\[
\omega_k^{(\text{zero})} = \omega_s / \cos[(2k - 1)\pi/(2N)] , \quad k = 1, 2, 3, \ldots, N_p, \quad (5.7)
\]

where \( N_p = (N+1)/2 \) if \( N \) is odd, and \( N_p = N/2 \) if \( N \) is even. Note that if \( N \) is odd the highest frequency where the response is zero is infinity: there are only \((N - 1)/2\) finite frequencies where the response is zero. If \( N \) is even there are \( N/2 \) finite frequencies where the response is zero.

Similarly, the attenuation equals \( A_s \) in the stopband when \( C_N^2(\omega_s/\omega) = 1 \). The frequencies of these minimum attenuation points may be found as follows:

\[
\omega_k^{(\text{min})} = \omega_s / \cos[(k\pi)/N] , \quad k = 1, 2, 3, \ldots, N_v, \quad (5.8)
\]

where \( N_v = (N-1)/2 \) if \( N \) is odd, and \( N_v = N/2 \) if \( N \) is even. Note that if \( N \) is even the highest frequency where the attenuation is equal to \( A_s \) is infinity.

The stopband response is denoted as “equiripple” since all of the stopband peaks (the points of minimum attenuation) are the same magnitude. It is noted that the frequency spacing between peaks are not equal: it is the magnitudes of the peaks that are equal.

The frequency at which the attenuation is equal to a given \( A_p \) may be found from (5.3):

\[
A_p = 10 \log \left( \frac{1 + \varepsilon^2 C_N^2(\omega_s/\omega_p)}{\varepsilon^2 C_N^2(\omega_s/\omega_p)} \right),
\]

and then solve for \( \omega_p \), making use of the hyperbolic form of (5.4):

\[
\omega_p = \omega_s / \cosh\left[\frac{(1/N) \cosh^{-1}(1/\sqrt{\varepsilon^{10} A_p/10 - 1})}{\varepsilon} \right]. \quad (5.9)
\]

See Figure 5.1 for plots of (5.3) for a normalized \( \omega_s \) of unity (\( A_p = 3 \) dB), \( A_s = 80 \) dB, and several values of \( N \). Recall that Butterworth and Chebyshev Type I filters both have magnitude frequency responses that monotonically decrease with increasing frequency throughout the transition band and the stopband. However, due to the rippling in the stopband, this is not the case for Chebyshev Type II filters, as can be seen in Figure 5.1. Also, as can be seen from (5.3), \( |H(j\omega)|^2 \rightarrow 0 \) as \( \omega \rightarrow \infty \).
if $N$ is odd, and is $\varepsilon^2/(1 + \varepsilon^2)$ if $N$ is even. Even though the frequency range does not go to infinity in Figure 5.1, this phenomenon is observable.

See Figure 5.2 for detailed plots of (5.3) across the passband. Note that the passband magnitude response is very flat. It is very comparable to the Butterworth passband magnitude response shown in Figure 3.2. In fact, for a large range of $\varepsilon$ it is superior: see Sections 5.8 and 5.9. In Figures 5.1 and 5.2, solid lines are for even orders, and dashed lines are for odd orders.
Example 5.1
Suppose \( N = 5 \), \( \omega_s = 1000 \text{ rad/s} \), and \( \varepsilon = 0.001 \) (\( A_s = 60 \text{ dB} \)), then, from (5.7), the frequencies where the magnitude frequency response is zero are 1051.46 rad/s, 1701.3 rad/s, and infinity. From (5.8) the frequencies where the attenuation in the stopband ripples to a minimum of are 1236.07 rad/s and 3236.07 rad/s. From (5.9), \( \omega_c = 417.39 \text{ rad/s} \).

5.2 FILTER SELECTIVITY AND SHAPING FACTOR

Applying (2.37), the definition of Filter Selectivity, to the square root of (5.3) results in

\[
F_s = \frac{\varepsilon N \omega_s}{\omega_c \sqrt{\omega_s^2 - \omega_c^2}} \frac{\sinh [N \cosh^{-1}(\omega_s/\omega_c)]}{(1 + \varepsilon^2 \cosh^2[N \cosh^{-1}(\omega_s/\omega_c)])^{3/2}}. \tag{5.10}
\]

If (5.9) is used, with \( A_p = 3 \text{ dB} \), and therefore \( \omega_p = \omega_c \), in (5.10) to eliminate any direct reference to \( \omega_s \), then (5.10) may be expressed as follows:

\[
F_s = \frac{\varepsilon N \cosh[(1/N) \cosh^{-1}(1/\varepsilon)] \sinh[\cosh^{-1}(1/\varepsilon)]}{2 \sqrt{2} \omega_c \sinh[(1/N) \cosh^{-1}(1/\varepsilon)]}. \tag{5.11}
\]

It is noted that (5.11) is identical to (4.9).

Let \( A \) be an arbitrary attenuation in dB relative to the DC value such that \( 0 < A \leq A_s \). From (5.3):

\[
A = 10 \log \left\{ \left[ 1 + \varepsilon^2 C_N^2(\omega_s/\omega) \right]/\varepsilon^2 C_N^2(\omega_s/\omega) \right\}. \tag{5.12}
\]

For a given \( A \), solving (5.12) for \( \omega \) would be equivalent to solving for the bandwidth at that attenuation \( A \):

\[
BW = \omega_s / \cosh \left[ (1/N) \cosh^{-1} \left\{ 1/\varepsilon \sqrt{10^{A/10} - 1} \right\} \right]. \tag{5.13}
\]

Using (5.13) and applying (2.38), the definition of Shaping Factor, the Chebyshev Type II filter Shaping Factor may be readily found:

\[
S_a^b = \frac{BW_b}{BW_a} = \frac{\cosh \left[ (1/N) \cosh^{-1} \left\{ 1/\varepsilon \sqrt{10^{A/10} - 1} \right\} \right]}{\cosh \left[ (1/N) \cosh^{-1} \left\{ 1/\varepsilon \sqrt{10^{B/10} - 1} \right\} \right]} \tag{5.14}
\]
Example 5.2

Suppose $a = 3 \text{ dB}, \ b = 80 \text{ dB}, \ \omega_r = 1$, and $\epsilon = 0.0001 \ (A_s = 80 \text{ dB})$. From (5.11), for $N = 1, 2, \cdots, 10$, $F_S$ may be computed to be 0.35, 0.71, 1.06, 1.43, 1.84, 2.28, 2.79, 3.35, 3.97 and 4.67 respectively. From (5.14), for $N$ from 1 through 10, $S^B_3$ may be computed to be 10000.0, 70.71, 13.59, 5.99, 3.69, 2.70, 2.18, 1.87, 1.67 and 1.53 respectively. □

5.3 DETERMINATION OF ORDER

An important step in the design of an analog filter is determining the minimum required order to meet given specifications. Refer to Figure 2.15 on page 52 in specifying the desired filter magnitude characteristics. As long as the filter magnitude frequency response fits within the acceptable corridor indicated in Figure 2.15, it satisfies the specifications.

Starting with (5.3):

$$-10 \log |H(j\omega_p)|^2 = A_p = 10 \log \left[ \frac{\epsilon^2 C_N^2(\omega_s/\omega_p) / \epsilon^2 C_N^2(\omega_s/\omega_p)}{\epsilon^2} \right]. \quad (5.15)$$

Temporarily let $\eta$, a real variable, assume the role of $N$, an integer, as is done in Chapters 3 and 4. Therefore, from (5.15):

$$\frac{1}{\epsilon^2} \cosh^2[\eta \cosh^{-1}(\omega_s/\omega_p)] = 10^{A_p/10} - 1,$$

from which, making use of (5.6),

$$\eta = \frac{\cosh^{-1} \left[ \sqrt{10^{A_p/10} - 1} / \sqrt{10^{A_p/10} - 1} \right]}{\cosh^{-1}(\omega_s/\omega_p)}.$$  

Letting $N = \lfloor \eta \rfloor$, where $\lfloor \eta \rfloor$ is the smallest integer equal to or larger than $\eta$ ($\eta < \lfloor \eta \rfloor < \eta + 1$), the minimum order required to meet the specifications may be determined from the following:

$$N = \left\lfloor \frac{\cosh^{-1} \left[ \sqrt{10^{A_p/10} - 1} / \sqrt{10^{A_p/10} - 1} \right]}{\cosh^{-1}(\omega_s/\omega_p)} \right\rfloor. \quad (5.16)$$

Note that (5.16) is identical to (4.14): for the same specifications, the minimum order required for a Chebyshev Type II filter is the same as that for a Chebyshev Type I filter.
Example 5.3

Suppose the following specifications are given: \( f_p = 3,000 \, \text{Hz} \), \( f_s = 7,000 \, \text{Hz} \), \( A_p = 2 \, \text{dB} \), and \( A_s = 60 \, \text{dB} \). From the right side of (5.16), \( \eta = 5.278 \). Therefore, \( N = 6 \).

5.4 INVERSE CHEBYSHEV POLYNOMIALS

A recursion for (5.4) may readily be developed, similar to that which was done in Section 4.4 for Chebyshev Type I filters. The resultant recursion is similar to (4.21):

\[
C_{N+1}(\omega_s/\omega) = 2 \left( \frac{\omega_s}{\omega} \right) C_N\left(\frac{\omega_s}{\omega}\right) - C_{N-1}(\omega_s/\omega). \tag{5.17}
\]

It is important to note that (5.17) applies equally to the cosine and hyperbolic forms of \( C_N(\omega_s/\omega) \).

If \( N = 0 \), and for convenience \( \omega_s \) is normalized to unity,

\[
\cos[N \cos^{-1}(1/\omega)] = \cosh[N \cosh^{-1}(1/\omega)] = 1 \quad \forall \omega,
\]

and therefore

\[
C_0(1/\omega) = 1 \quad \forall \omega.
\]

If \( N = 1 \),

\[
\cos[N \cos^{-1}(1/\omega)] = \cosh[N \cosh^{-1}(1/\omega)] = 1/\omega,
\]

and therefore

\[
C_1(1/\omega) = 1/\omega \quad \forall \omega.
\]

For \( N > 1 \) the recursion (5.17) may be used. For \( N = 2 \),

\[
C_2(1/\omega) = (2/\omega) C_1(1/\omega) - C_0(1/\omega) = (2/\omega^2) - 1.
\]

Several Chebyshev polynomials are shown in Table 4.2. Inverse Chebyshev polynomials may be obtained from Table 4.2 by replacing \( \omega \) with \( 1/\omega \). Note that if \( \omega_s \) is not normalized to unity, the inverse Chebyshev polynomial is as shown in Table 4.2 with \( \omega \) replaced by \( \omega_s/\omega \).

It can be shown that the square of all inverse Chebyshev polynomials have only even powers of \( \omega \), and that multiplied by \( e^2 \) and added to unity they have no real roots. If they are not added to unity, as appears in the numerator of (5.3), then they do have real roots: those roots may be found by (5.7). Therefore, the Analog Filter Design Theorem is satisfied. For example, see Example 5.4.
Example 5.4  
Suppose \( N = 3 \), \( \omega_s = 100 \), and \( \varepsilon = 0.01 \). Then  
\[
C_3(100/\omega) = 4 \left(\frac{100}{\omega}\right)^3 - 3 \left(\frac{100}{\omega}\right),
\]
and  
\[
|H(j\omega)|^2 = \frac{9 \omega^8 - 2.4 \times 10^5 \omega^5 + 1.6 \times 10^9}{\omega^6 + 9 \omega^4 - 2.4 \times 10^5 \omega^2 + 1.6 \times 10^9}.
\]
A root of the denominator requires that  
\[
2.4 \times 10^5 \omega^2 = \omega^6 + 9 \omega^4 + 1.6 \times 10^9,
\]
which has no real solution, i.e., no real roots. The numerator has one real repeated root, which is 115.47. Therefore, the Analog Filter Design Theorem is satisfied: there is a corresponding \( H(s) \) that meets all of the imposed constraints of Section 2.6. Therefore a circuit can be implemented with the desired third-order Chebyshev Type II response.

5.5 LOCATION OF THE POLES AND ZEROS

Starting with (5.3) and following the procedure used in Section 2.7:  
\[
Y(s) = H(s)H(-s) = \frac{\varepsilon^2 C_N^2(\omega_s/-js)}{1 + \varepsilon^2 C_N^2(\omega_s/-js)}.  \tag{5.18}
\]
The zeros of (5.18) may be found by setting  
\[
C_N(\omega_s/-js_k) = 0,
\]
and solving for the values of \( s_k \). The trigonometric cosine form of (5.4) must be used since the inverse hyperbolic cosine of zero doesn’t exist:  
\[
N \cos^{-1}(\omega_s/-js_k) = (\pi/2)(2k - 1), \quad k = 1, 2, 3, \ldots. \tag{5.19}
\]
Solving (5.19) for \( s_k \) results in  
\[
s_k = \pm j\omega_k^{(\text{zero})}, \quad k = 1, 2, 3, \ldots, N_p, \tag{5.20}
\]
Section 5.5 Location of the Poles and Zeros
where $\omega_k^{(zero)}$ is given by (5.7).

The poles of (5.18) may be found by setting

$$C_N(\omega_s / -js_k) = \pm j/\varepsilon.$$ \hfill (5.20)

Since $\varepsilon < 1$, then $|\pm j/\varepsilon| > 1$, and the hyperbolic form of $C_N$ is perhaps the more appropriate:

$$C_N(\omega_s / -js_k) = \cosh[N \cosh^{-1}(\omega_s / -js_k)] = \pm j/\varepsilon.$$

As noted in Chapter 4, since $\pm j/\varepsilon$ is complex, either form, the cosine or the hyperbolic, is equally valid. Either approach will yield the same equations for finding the poles.

Since the approach here is identical to the approach used in Section 4.5, except that $-js_k/\omega_p$ is replaced by $\omega_s / -js_k$, it follows that the equivalent to (4.32) would be as follows:

$$s_k = j\omega_s / \cosh[(1/N) \sinh^{-1}(1/\varepsilon) + j(\pi/[2N])(2k - 1)]$$

$$= \omega_s / \left\{-\sinh[(1/N) \sinh^{-1}(1/\varepsilon)] \sin[(\pi/[2N])(2k - 1)]
+ j \cosh[(1/N) \sinh^{-1}(1/\varepsilon)] \cos[(\pi/[2N])(2k - 1)]\right\}. \hfill (5.21)$$

For left-half plane poles:

$$s_k = \sigma_k + j\omega_k,$$
$$\sigma_k = -\omega_s \sinh[(1/N) \sinh^{-1}(1/\varepsilon)] \sin[(\pi/[2N])(2k - 1)]/D(k),$$
$$\omega_k = -\omega_s \cosh[(1/N) \sinh^{-1}(1/\varepsilon)] \cos[(\pi/[2N])(2k - 1)]/D(k), \hfill (5.22)$$

$$k = 1, 2, 3, \ldots, N,$$

where

$$D(k) = \sinh^2[(1/N) \sinh^{-1}(1/\varepsilon)] \sin^2[(\pi/[2N])(2k - 1)]$$

$$+ \cosh^2[(1/N) \sinh^{-1}(1/\varepsilon)] \cos^2[(\pi/[2N])(2k - 1)].$$

It is interesting to note that the poles of the Chebyshev Type I filter, for a normalized $\omega_p = 1$, may be expressed as follows:

$$s_k^{(norm)} = \sqrt{D(k)} \ e^{j\theta_k},$$

where

$$\theta_k = -\tan^{-1}\left\{\frac{\cosh[(1/N) \sinh^{-1}(1/\varepsilon)] \cos[(\pi/[2N])(2k - 1)]}{\sinh[(1/N) \sinh^{-1}(1/\varepsilon)] \sin[(\pi/[2N])(2k - 1)]}\right\},$$

Chapter 5  Chebyshev Type II Filters
and the poles of the Chebyshev Type II filter, for a normalized $\omega_s = 1$, may be expressed as follows:

$$s_k^{II(n)} = \frac{1}{\sqrt{D(k)}} e^{-j\theta_k},$$

and therefore, the $s_k^{II(n)}$ poles are the $s_k^{I(n)}$ poles reflected about a unit circle in the $s$ plane: the magnitudes are inversely related and the phase angles have opposite polarity. This inverse relationship between the normalized poles of the Chebyshev Type I transfer function and those of the Chebyshev Type II is often noted in the literature, and is implied, as noted above, by contrasting (4.33) and (5.22). It is possible to find the poles of a Chebyshev Type II transfer function by first finding the poles of a Chebyshev Type I transfer function and then converting them into Chebyshev Type II poles. However, this relationship between normalized poles doesn’t seem to yield any practical advantage. Direct use of (5.22) is quite adequate for finding the poles of a Chebyshev Type II transfer function. Also note, by comparing Tables 4.1 and 5.1, that practical values for $\epsilon$ are very different for the two transfer functions.

**Example 5.5**

Suppose $N = 4$, $\omega_s = 1000 \text{ rad/s}$, and $\epsilon = 0.01$. From (5.9),

$$\omega_c = 496.71.$$  

From (5.22), the poles are

$$s_k = -171.16 \pm j\,476.10, \quad -504.53 \pm j\,240.79.$$  

From (5.20), the zeros are

$$s_k = \pm j\,1082.39, \quad \pm j\,2613.13.$$  

From the poles and zeros, and noting that the DC gain is unity:

$$H(s) = \frac{0.01 \left(s^2 + 1,171,572.88\right) \left(s^2 + 6,828,427.13\right)}{\left(s^2 + 342.32\,s + 255,963.37\right) \left(s^2 + 1,009.06\,s + 312,529.09\right)} ,$$

or,

$$H(s) = \frac{0.01 \left(s^4 + 8 \times 10^6\,s^2 + 8 \times 10^{12}\right)}{s^4 + 1,351.38\,s^3 + 913,911.63\,s^2 + 365,266,792.0\,s + 8 \times 10^{10}}.$$  

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**Section 5.5 Location of the Poles and Zeros**
5.6 PHASE RESPONSE, PHASE DELAY, AND GROUP DELAY

A Chebyshev Type II filter, as seen above, is designed to meet given magnitude response specifications. Once the transfer function is determined, it may be put in the following form:

\[ H(s) = \frac{K \sum_{k=0}^{M} b_k s^k}{\sum_{i=0}^{N} a_i s^i}, \tag{5.23} \]

which is of the form of \(2.39\). Given \(5.23\), the phase response, from \(2.79\), may be stated as follows:

\[ \angle H(j\omega) = \arctan \left( \frac{R_d(\omega) I_n(\omega) - R_n(\omega) I_d(\omega)}{R_n(\omega) R_d(\omega) + I_n(\omega) I_d(\omega)} \right), \tag{5.24} \]

where \(R_d(\omega)\) and \(I_d(\omega)\) denote the real and imaginary parts of the denominator, respectively, and \(R_n(\omega)\) and \(I_n(\omega)\) denote the real and imaginary parts of the numerator of \(5.23\) evaluated with \(s = j\omega\).

The phase response of a Chebyshev Type II filter, with a normalized \(\omega_c = 1\), a somewhat arbitrary, but common value of \(A_s = 80 \, \text{dB} \ (\varepsilon = 0.0001)\), and several values of \(N\), is shown in Figure 5.3. The phase response, from \(\omega = 0\) until the first phase discontinuity, which occurs at \(\omega = 1.55 \, \text{rad/s}\) for the tenth-order response,

![Figure 5.3](image_url)

**Figure 5.3** A plot of the phase response for a Chebyshev Type II filter with normalized \(\omega_c = 1\), \(A_s = 80 \, \text{dB}\), and for values of \(N\) from 1 through 10.
is the total phase, in contrast to the principal phase. The total phase, as shown for Butterworth filters in Figure 3.6, and for Chebyshev Type I filters in Figure 4.4, and and for Chebyshev Type II filters until the first phase discontinuity in Figure 5.3, is important because phase delay and group delay are directed related to the total phase response. Each of the phase discontinuities seen in Figure 5.3 are $\pi \ rad$. The phase response in Figure 5.3 for $\omega$ beyond, and including, the first phase discontinuity is not total phase, but rather pseudo-principal phase. That is, the phase shown is the total phase plus $m \ 2\pi \ rad$, where $m$ is an integer. This technique allows for a less congested set of plots that is easier to read. In fact, each of the phase discontinuities, if total phase was to be preserved, are $+\pi \ rad$. The phase discontinuities occur at transmission zeros, which are on the $j\omega$ axis; as $\omega$ increases through a zero, the phase response encounters a $+\pi \ rad$ discontinuity. It is interesting to note that while Butterworth and Chebyshev Type I filters each have a phase response in the limit, as $\omega$ approaches infinity, of $-N\pi/2 \ rad$, this is not true for Chebyshev Type II filters. Due to the finite transmission zeros, the phase response in the limit, as $\omega$ approaches infinity, for Chebyshev Type II filters, is zero, for $N$ even, and is $-\pi/2 \ rad$, for $N$ odd.

Taking the initial phase slope as a linear-phase reference, deviations from linear phase, for a normalized $\omega_c = 1$, $A_s = 80 \ dB$, and for several values of $N$, are shown in Figure 5.4. In the figure, solid lines are for even orders, and dashed lines are for odd orders. The phase deviation is shown in the figure from $\omega = 0$ until just before the first phase discontinuity occurs. Each phase discontinuity causes a $+\pi$ discontinuity in the phase deviation, but if plotted is somewhat misleading.

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Figure 5.4 Phase deviation from linear for a Chebyshev Type II filter with normalized $\omega_c = 1$, $A_s = 80 \ dB$, and for values of $N$ from 1 through 10.

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1The principal phase is restricted to the range of $\pm \pi$. 

Section 5.6 Phase Response, Phase Delay, and Group Delay
since the magnitude response is zero at the same frequency and is, in general, in the stopband. Over the frequency range of the figure, $0 \leq \omega \leq 2 \text{ rad/s}$, phase discontinuities only effect the plots for orders 8, 9, and 10, as can be seen in the figure.

The phase delay, $t_{pd}(\omega)$, for a filter is defined in (2.80), which is repeated here for convenience:

$$t_{pd}(\omega) = - \frac{\angle H(j\omega)}{\omega}.$$  (5.25)

The group delay for a filter, $t_{gd}(\omega)$, is defined by (2.81) and is repeated here for convenience:

$$t_{gd}(\omega) = - \frac{d\angle H(j\omega)}{d\omega}.$$  (5.26)

The phase delay of a Chebyshev Type II filter, with a normalized $\omega_c = 1$, $A_g = 80 \text{ dB}$, and for several values of $N$, is shown in Figure 5.5. Note that the phase delay discontinuities occur at the frequencies where there are phase discontinuities, that is, at the frequencies of transmission zeros. Since each phase discontinuity is $+\pi \text{ rad}$, each phase delay discontinuity is $-\pi/\omega \text{ sec}$, where $\omega$ is the frequency of the discontinuity. The effect, therefore, of the transmission zeros, is that the phase delay for frequencies well beyond the passband approaches zero much more rapidly than it does for either Butterworth or Chebyshev Type I filters (see Figures 3.8 and 4.6).

The group delay is shown in Figure 5.6. Note that the group delay values at DC are very close to the phase delay values at DC, and that for all $\omega < 0.5$ the phase delay and the group delay are comparable. However, as $\omega$ approaches the 3

![Figure 5.5](image)

**Figure 5.5** A plot of the phase delay for a Chebyshev Type II filter with normalized $\omega_c = 1$, $A_g = 80 \text{ dB}$, and for values of $N$ from 1 through 10.
$dB$ corner frequency, in this case unity, the group delay becomes large due to the nonlinearity of the phase response near the corner frequency (see Figures 5.3 and 5.4). Note the points of discontinuity in the group delay, for example at $\omega = 1.55 \text{ rad/s}$ for $N = 10$. As can be seen from (5.26), at each $+\pi \text{ rad}$ phase discontinuity the group delay is theoretically $-\infty \text{ s}$. That is, the group delay is theoretically an infinite time advance, rather than a delay, at the point of a phase discontinuity; however, since this occurs only at a point along the frequency axis, and at a point of a transmission zero, the filter magnitude response at that point is zero; there is nothing to advance. In Figure 5.6, these points of infinite time advance are plotted with non-zero width; this is a result of the plotting software and from the fact that the calculation frequency-sample width is non-zero. Also, for convenience, the minimum delay value of the figure is zero.

5.7 TIME-DOMAIN RESPONSE

The unit impulse response of a Chebyshev Type II filter, with a normalized $\omega_c = 1$, $A_s = 80 \text{ dB}$, and for several values of $N$, is shown in Figure 5.7. Note that for even orders there is an impulse at the origin (not shown in the figure), but the weight of these impulses is $\epsilon$ (in this case, since $A_s = 80 \text{ dB}$, $\epsilon = 0.0001$), and therefore these impulses are insignificant. The unit step response of a Chebyshev Type II filter, with normalized $\omega_c = 1$, $A_s = 80 \text{ dB}$, and for several values of $N$, is shown in Figure 5.8.
There are several ways in which Chebyshev Type II filters may be compared with Butterworth and Chebyshev Type I filters. The magnitude frequency responses may be compared, the phase responses, the phase delays, the group delays, the unit impulse responses, and the unit step responses, for example.

**Figure 5.7** A plot of the unit impulse response for a Chebyshev Type II filter with normalized $\omega_c = 1$, $A_s = 80$ dB, and for values of $N$ from 2 through 10.

**5.8 COMPARISON WITH BUTTERWORTH AND CHEBYSHEV TYPE I FILTERS**

There are several ways in which Chebyshev Type II filters may be compared with Butterworth and Chebyshev Type I filters. The magnitude frequency responses may be compared, the phase responses, the phase delays, the group delays, the unit impulse responses, and the unit step responses, for example.

**Figure 5.8** A plot of the unit step response for a Chebyshev Type II filter with normalized $\omega_c = 1$, $A_s = 80$ dB, and for values of $N$ from 1 through 10.
It is noted that, for a given set of filter specifications, the minimum order required for a Chebyshev Type II response will never be greater than that required for a Butterworth response: it will frequently be less. It is not always less, since the orders are restricted to integers. As noted in Section 5.3, the minimum order required for a Chebyshev Type II filter is identical with that for a Chebyshev Type I filter.

Although the passband of a Chebyshev Type II response was not designed to be maximally flat in any sense, yet, as can be seen by comparing Figures 5.2 and 3.2, the passband magnitude response of a Chebyshev Type II filter is comparable with that of a Butterworth. In fact, over a wide range of filter specifications, the passband magnitude response of a Chebyshev Type II filter, with the same order and same 3 dB corner frequency, is more flat than that of a Butterworth filter. One way of demonstrating this is by comparing Filter Selectivity for the two filters. Let (5.11) be denoted $F_S^{(CII)}$ and (3.7) be denoted $F_S^{(B)}$. It can be shown that

\[
F_S^{(CII)} \geq F_S^{(B)}
\]

(5.27)

for all $\varepsilon$, $N$, and $\omega_\varepsilon$. Equality in (5.27) is achieved only for $N = 1$ or for very small values of $\varepsilon$. For example, for $\varepsilon = 0.0001$, $N = 10$, and $\omega_\varepsilon = 1$, then

\[
F_S^{(CII)} = 4.667, \quad F_S^{(B)} = 3.536.
\]

It can be shown that, for all practical values of $\varepsilon$, that the Shaping Factor of a Chebyshev Type II filter is always less than that for a Butterworth filter of the same order. To illustrate this, compare the results of Example 5.2 to that of Example 3.1; in both cases $a = 3$ dB and $b = 80$ dB. For example, for $N = 8$, $\Delta_3^{80}$ for the Butterworth filter is 3.16, whereas it is only 1.87 for the Chebyshev Type II filter, which indicates that the attenuation in the transition band increases more rapidly as a function of $\omega$ for the Chebyshev Type II filter than it does for the Butterworth. This, of course, can readily be observed by comparing Figures 3.1 and 5.1.

The above may imply that a Chebyshev Type II filter is superior to a Butterworth filter of the same order. However, the phase response of a Butterworth filter is more nearly linear than is that of a Chebyshev Type II filter of the same order. The differences, however, are not great. The phase deviation from linear for a Chebyshev Type II filter, as shown in Figure 5.4, is greater than it is for a Butterworth filter, as shown in Figure 3.7. This is, of course, reflected in the phase delay and the group delay, but comparing Figure 5.5 with 3.8, and 5.6 with 3.9, shows that the differences are not great. Comparing plots of the unit impulse response and the unit step response of the two filters also shows that the differences are not great.

The comparison between Chebyshev Type II filters and Chebyshev Type I filters is more objective and straight-forward, especially since the minimum required

Section 5.8 Comparison with Butterworth and Chebyshev Type I Filters
order to meet design specifications is identical for both filters. A Chebyshev Type II filter, of the same order, has a more constant magnitude response in the passband, a more nearly linear phase response, a more nearly constant phase delay and group delay, and less ringing in the impulse and step responses, than does a Chebyshev Type I filter: compare Figures 5.2 and 4.2, 5.4 and 4.5, 5.5 and 4.6, 5.6 and 4.7, 5.7 and 4.8, and 5.8 and 4.9. However, while (5.11) and (4.9), equations for Filter Selectivity, are the same, the numerical values for \( \varepsilon \) differ significantly for the two filters. For all practical filters, \( \varepsilon \) for the Chebyshev Type II filter will be significantly smaller than that for the comparable Chebyshev Type I filter. To illustrate this, compare Table 5.1 to Table 4.1. The result is that, for all practical filters, \( F_S \) for the Chebyshev Type I filter will be significantly larger than for the comparable Chebyshev Type II filter: compare Figures 5.2 and 4.2. However, the comparison of Shaping Factor values is not consistent. It is possible for a Chebyshev Type II filter to have a smaller \( S_a^b \) than a corresponding Chebyshev Type I filter. To illustrate this, compare 10th-order responses in Figures 5.1 and 4.1 with \( a = 3 \text{ dB} \) and \( b = 80 \text{ dB} \).

5.9 CHAPTER 5 PROBLEMS

5.1 Given the defining equations for a Chebyshev Type II response, (5.3) and (5.4), and given that \( \omega_c = 1000 \text{ rad/s}, \ A_s = 50 \text{ dB} \), and \( N = 3 \):
(a) Determine the value of \( \varepsilon \).
(b) Determine the value of \( \omega_s \).
(c) Determine the frequencies where the response is zero.
(d) Determine the frequencies in the stopband where the attenuation is \( A_s \).
(e) Accurately sketch the magnitude frequency response. Use only a calculator for the necessary calculations. Use a vertical scale in dB (0 to -60 dB), and a linear radian frequency scale from 0 to 5000 rad/s.
(f) Accurately sketch the magnitude frequency response. Use only a calculator for the necessary calculations. Use a linear vertical scale from 0 to 1, and a linear radian frequency scale from 0 to 2000 rad/s.

5.2 Given the defining equations for a Chebyshev Type II response, (5.3) and (5.4), and given that \( \omega_c = 1000 \text{ rad/s}, \ A_s = 60 \text{ dB} \), and \( N = 6 \):
(a) Determine the value of \( \varepsilon \).
(b) Determine the value of \( \omega_s \).
(c) Determine the frequencies where the response is zero.
(d) Determine the frequencies in the stopband where the attenuation is \( A_s \).
(e) Accurately sketch the magnitude frequency response. Use only a calculator for the necessary calculations. Use a vertical scale in dB (0
5.3 Starting with (2.37) and the square root of (5.3), derive (5.11).

5.4 On page 159 it is mentioned that for a large range of $\varepsilon$ the passband magnitude response for a Chebyshev Type II filter is more flat than that of a Butterworth filter of the same order. On page 171 this concept is expanded upon. Since both responses are relatively flat in the passband, if they both have the same $\omega_c$, then the one with the larger Filter Selectivity implies that the magnitude frequency response remains closer to unity as $\omega$ approaches $\omega_c$ than does the other one. Verify that, for the same order and the same $\omega_c$, Filter Selectivity for a Chebyshev Type II filter is greater than or equal to Filter Selectivity for a Butterworth filter, and that they are equal only for $N = 1$ or for very small values of $\varepsilon$. That is, verify (5.27).

5.5 Determine the value of Filter Selectivity for the Chebyshev Type II filter specified in Problem 5.2. Compare this value with the Filter Selectivity for a Butterworth filter with similar specifications: $\omega_c = 1000 \text{ rad/s}$, and $N = 6$. Compare this value with the Filter Selectivity for a Chebyshev Type I filter with similar specifications: $\omega_c = 1000 \text{ rad/s}$, $A_p = 1.5 \text{ dB}$, and $N = 6$.

5.6 Determine the value of the Shaping Factor for the Chebyshev Type II filter specified in Problem 5.2, for $a = 3 \text{ dB}$ and $b = 60 \text{ dB}$. Compare this value with the Shaping Factor for a Butterworth filter with similar specifications: $\omega_c = 1000 \text{ rad/s}$, and $N = 6$. Compare this value with the Shaping Factor for a Chebyshev Type I filter with similar specifications: $\omega_c = 1000 \text{ rad/s}$, $A_p = 1.5 \text{ dB}$, and $N = 6$.

5.7 Estimate the Shaping Factor for a Chebyshev Type II filter with $N = 5$, $\omega_c = 1$, and $A_p = 80 \text{ dB}$, from Figure 5.1. Compare your results with that obtained from (5.14).

5.8 Suppose filter specifications are stated as follows: $f_c = 3500 \text{ Hz}$, $f_s = 7000 \text{ Hz}$. Determine the required filter order to meet the given specifications:

(a) For a Chebyshev Type II filter with $A_p = 40 \text{ dB}$.
(b) For a Chebyshev Type II filter with $A_p = 60 \text{ dB}$.
(c) For a Chebyshev Type II filter with $A_p = 80 \text{ dB}$.
For a Chebyshev Type II filter with $A_s = 100$ dB.

For comparison purposes, repeat parts (a) through (d) for a Butterworth filter.

For comparison purposes, repeat parts (a) through (d) for a Chebyshev Type I filter with 0.1 dB of ripple.

For comparison purposes, repeat parts (a) through (d) for a Chebyshev Type I filter with 0.5 dB of ripple.

For comparison purposes, repeat parts (a) through (d) for a Chebyshev Type I filter with 1.5 dB of ripple.

Given that $N = 4$, $\omega_s = 100$, and $\varepsilon = 0.01$, express $|H(j\omega)|^2$ in polynomial form similar to Example 5.4, and demonstrate that it satisfies the Analog Filter Design Theorem.

Prove that the magnitude-squared frequency response of Problem 5.1 satisfies the Analog Filter Design Theorem.

Prove that the magnitude-squared frequency response of Problem 5.2 satisfies the Analog Filter Design Theorem.

Determine the 9-th and 10-th order inverse Chebyshev polynomials.

Sketch the square of the 4-th order inverse Chebyshev polynomial for $\omega$ from 0 to 1.1 rad/s. Compute the square of (5.4) over this same radian frequency range for $\omega_s = 1$, and verify that it is numerically the same as the inverse Chebyshev polynomial.

Determine the poles and zeros of the filter specified in Problem 5.8 (a).

Determine the poles and zeros of the filter specified in Problem 5.8 (d).

Determine the poles and zeros of the filter specified in Problem 5.9.

Suppose $N = 4$, $\omega_s = 1000$ rad/s, and $\varepsilon = 0.01$. Determine the transfer function $H(s)$. That is, verify the results of Example 5.5.

Determine the transfer function $H(s)$ for the Chebyshev Type II filter specified in Problem 5.1. Express the denominator of $H(s)$ in two ways: (1) As a polynomial in $s$. (2) As the product of a second-order polynomial in $s$, the roots of which being complex conjugates, and a first-order term. State the numerical values of the three poles and the finite-value zeros. Sketch the six poles and the zeros of $H(-s)H(s)$ on an $s$ plane.
5.19 Determine the transfer function $H(s)$ for the Chebyshev Type II filter specified in Problem 5.2. Express the denominator of $H(s)$ in two ways: (1) As a polynomial in $s$. (2) As the product of three second-order polynomials in $s$, the roots of each second-order polynomial being complex conjugates. Express the numerator of $H(s)$ in two ways: (1) As a polynomial in $s$. (2) As the product of three second-order polynomials in $s$, the roots of each second-order polynomial being complex conjugates. State the numerical values of the six poles and six zeros. Sketch the twelve poles and zeros of $H(-s)$ $H(s)$ on an $s$ plane.

5.20 Suppose a Chebyshev Type II filter is be used that has a minimum attenuation of 60 dB at 4000 Hz ($A_s = 60$ dB).

(a) If $f_p = 3000$ Hz and $A_p = 1$ dB, what is the minimum order required?
(b) If $A_p = 1$ dB and $N = 10$, what is the maximum value of $f_p$?
(c) If $f_p = 3000$ Hz and $N = 10$, what is the minimum value of $A_p$?

5.21 Under the conditions of part (c) of Problem 5.20, determine the transfer function $H(s)$, and give numerical values for all the poles and zeros.

5.22 Sketch the step response of a 10-th order Chebyshev Type II filter with $f_c = 1000$ Hz and $A_s = 80$ dB. Refer to Figure 5.8 and make use of the scaling property of Fourier transforms. What would the maximum group delay be for this filter, and at what frequency would it occur? At what time would the peak of the unit impulse response of this filter be, and what would be the value of that peak?

5.23 Using the MATLAB functions cheb2ap, impulse and step:

(a) Determine the transfer function in polynomial form, and also factored to indicate the poles and zeros, of a Chebyshev Type II filter with $\omega_p = 1$, $A_p = 1.2$ dB, and $N = 6$.
(b) Determine the impulse response and the step response for the filter of part (a).
(c) By multiplying the pole vector and the zero vector found in part (a) by $2\pi 1000$ determine the transfer function of a Chebyshev Type II filter with $f_p = 1000$ Hz, $A_p = 1.2$ dB, and $N = 6$.
(d) Determine and plot the magnitude frequency response of the filter of part (c) by using the MATLAB function freqs. Use a vertical scale in dB and a linear horizontal scale from 0 to 5000 Hz. Also determine and plot the phase response over this same frequency range. Use the MATLAB function unwrap rather than plotting the principle phase.
(e) By appropriately scaling the impulse response and the step response of
part (b), determine and plot the impulse response and the step response of the filter of part (c). That is, the time axis for the step response needs to be scaled by $1/(2\pi 1000)$, and the unit impulse response needs the same time-axis scaling and requires an amplitude scaling of $2\pi 1000$.

(f) Determine and plot the phase delay of the filter of part (c). Note that this is easily obtained from the phase response of part (d).

(g) Determine and plot the group delay of the filter of part (c). Note that this also is easily obtained from the phase response of part (d):

$$t_{\phi}(n) \equiv -(\phi(n) - \phi(n-1))/S'_d,$$

where $\phi(n)$ is the phase in radians at step $n$, and $S'_d$ is the step size in rad/s.

5.24 To demonstrate the critical nature of a filter design, this problem experiments some with the filter of Problem 5.23 (c). Multiply the highest-frequency pole-pair of the filter of Problem 5.23 (c) by 1.1, leaving all other poles and zeros unchanged. Determine and plot the magnitude frequency response of the filter by using the MATLAB function freqs. Use a vertical scale in dB and a linear horizontal scale from 0 to 5000 Hz. Also determine and plot the phase response over this same frequency range. Use the MATLAB function unwrap rather than plotting the principle phase. Compare these results with that obtained for Problem 5.23 (d).

5.25 This problem continues to demonstrate the critical nature of a filter design, and is a continuation of Problem 5.24. Multiply the lowest-frequency zero-pair of the filter of Problem 5.23 (c) by 1.1, leaving all other poles and zeros unchanged. Determine and plot the magnitude frequency response of the filter by using the MATLAB function freqs. Use a vertical scale in dB and a linear horizontal scale from 0 to 5000 Hz. Also determine and plot the phase response over this same frequency range. Use the MATLAB function unwrap rather than plotting the principle phase. Compare these results with that obtained for Problem 5.23 (d).